THE NON-SINGULAR CUBIC SURFACES

THE NON-SINGULAR CUBIC SURFACES

A New Method of Investigation
with Special Reference
to Questions of Reality

BY

B. SEGRE

OXFORD
AT THE CLARENDON PRESS
1942

OXFORD UNIVERSITY PRESS
AMEN HOUSE, E.C. 4
London Edinburgh Glasgow New York
Toronto Melbourne Capetown Bombay
Calcutta Madras
HUMPHREY MILFORD
PUBLISHER TO THE UNIVERSITY

PRINTED IN GREAT BRITAIN AT THE UNIVERSITY PRESS, OXFORD
BY JOHN JOHNSON, PRINTER TO THE UNIVERSITY

PREFACE

The study of the general cubic surface dates from 1849, in which year the 27 lines were found by Cayley and Salmon; the discovery of the Sylvester pentahedron followed two years later, and the theory attained a remarkable degree of elegance and completeness with the introduction of the plane representation, made independently by Clebsch and Cremona in 1866. Nevertheless, even to-day the study of what Cayley has called 'the complicated and many-sided symmetry' of the relations between the 27 lines cannot be said to have been exhausted, especially in the real domain, where group properties have not yet been considered; nor does Schläfli's well-known notation for the lines readily lend itself to the description and deeper investigation of that symmetry.

In the present work a new method allows us to examine the whole theory in a suggestive manner, and to simplify and complete it at various essential points. This method makes a systematic use of the principle of continuity, as well as of the consideration of the limit of the 27 lines when the cubic surface degenerates into three planes, and leads to a new and useful notation and plane representation for the 27 lines.† By means of it we obtain, in Ch. I, a complete picture of the more important properties—only a few of which are essentially new—of the 27 lines and the 45 tritangent planes, such as those relating to the double-sixes, to the Steiner trihedra, etc., considered in the complex domain. The study of the symmetry and the relations between these figures is completed in Ch. II by the examination of their group properties, connected with the group of order 51840 of the 27 lines.

Chapter III—the longest and most characteristic of the present work—is devoted entirely to the real domain. In this chapter it is shown, by simple considerations of continuity, that the real, non-singular cubic surfaces are distributed in five connected continuous systems, and the

† Considerations of continuity in the study of the cubic surface have previously been adopted by various authors, especially by Klein, who, in an important memoir, obtained some interesting properties of the surface in the real domain from the skilful examination of particular models (four-nodal and diagonal surfaces); cf. F. Klein, 'Ueber Flächen dritter Ordnung', *Math. Ann.*, vol. 6 (1873), pp. 551–81. An alternative method is offered by Geiser's stereographic projection: it was used by Zeuthen with great success to deduce properties of the cubic surface from those of the plane quartic curve: cf. H. G. Zeuthen, 'Etudes des propriétés de situation des surfaces cubiques', *Math. Ann.*, vol. 8 (1875), pp. 1–30. Cf. also, for further references to the very extensive literature on our subject, A. Henderson, 'The Twenty-seven Lines upon the Cubic Surface', *Cambridge Tract*, No. 13 (Cambridge University Press, 1911); L. Berzolari, 'Flächen dritter Ordnung', Pascal's *Repertorium*, ii. 2 (Leipzig, Teubner, 1922), pp. 783–849; W. Fr. Meyer, 'Flächen dritter Ordnung', *Encycl. der Math. Wiss.* iii. C 10a, pp. 1437–531.

contiguity relations between these systems are then studied. We thus obtain the five types of Schläfli's classification. For each type a great number of properties of the configurations formed by the 27 lines, especially their group properties, are investigated, in the real field, by means of the graphical representation; in particular, we carry out the classification and construction of the real double-sixes of the different types. Closely related to all this is the study, essentially topological in character, of the generalized polyhedron determined on a real cubic surface by its real lines, which leads to a clear understanding of the form of such surfaces and their dual surfaces, to various properties of their parabolic curves and to their classification in the affine space.

The fourth and last chapter, unlike the preceding, is predominantly analytical. Here, after proving Sylvester's theorem, we determine the non-singular cubic surfaces for which the reduction to Sylvester's canonical form is either not unique or not possible; in such cases we give the appropriate canonical forms. Then follows the determination of all the cubic surfaces (non-singular or singular) whose Hessians are reducible, and a proof of the property that any quaternary cubic form can be expressed as the sum of at most seven cubes. As an application of the canonical equations, we study and classify completely, in the complex as well as in the real field, the non-singular cubic surfaces which possess homographic self-transformations; such surfaces are, with a single exception in the real field, identical with the surfaces possessing Eckardt points, i.e. simple points at which three lines of the surface concur. Finally, we indicate how, in the real field, the type of a non-singular cubic surface can be deduced from its canonical equation.

Further applications of the degeneration method to the study of general cubic forms in spaces of three and four dimensions are briefly pointed out in two of the three appendixes which end the volume.

I wish to express my deepest gratitude to the 'Society for the Protection of Science and Learning' and to the British people whose generous support has enabled me to continue mathematical work.

I am under great obligations for the printing of this book to Messrs. L. Roth and D. B. Scott, who have carefully read the manuscript and improved my English, to Dr. J. A. Todd, who has helped in reading the proofs, and to the staff of the Clarendon Press, whose invaluable co-operation at this time of difficulties I have particularly appreciated.

B. S.

CAMBRIDGE,
December 1941.

CONTENTS

I. THE 27 LINES AND ONE OF THEIR SIMPLEST DEGENERATIONS

II. THE GROUP $\mathfrak{G}$ OF THE 27 LINES

III. THE REAL CUBIC SURFACES

IV. THE SYLVESTER PENTAHEDRON AND THE CUBIC SURFACES WITH HOMOGRAPHIC TRANSFORMATIONS INTO THEMSELVES

XIII: SYLVESTER'S CANONICAL FORM AND EXCEPTIONAL CUBIC SURFACES

XIV: AUTOPROJECTIVE NON-SINGULAR CUBIC SURFACES

XV: THE TYPE OF A REAL NON-SINGULAR CUBIC SURFACE DEDUCED FROM ITS CANONICAL EQUATION

APPENDIX I

APPENDIX II

APPENDIX III

I

THE 27 LINES AND ONE OF THEIR SIMPLEST DEGENERATIONS

I. The 27 lines of a non-singular cubic surface degenerating into three planes

1. The cone circumscribed to a non-singular cubic surface F from a generic point P is of order 6, class 12, and has no ordinary double generators; in virtue of one of the Plücker formulae it has consequently 27 bitangent planes, i.e. there are 27 bitangent planes of F through P. Since a bitangent plane of F contains a line of this surface and conversely, it follows that:

A non-singular cubic surface contains 27 *lines.*

A necessary and sufficient condition that a line r of a cubic surface F should absorb (at least) two of its 27 lines is that F has a double point on r; so that *the* 27 *lines of F are all distinct, if F is non-singular.* We observe, in fact, that two cases only can arise if there is a line of F infinitely near to r, according as this line is incident or skew to r; in the first case F has a fixed tangent plane along r, and in the second the tangent planes correspond homographically to their points of contact. A variable plane through r intersects F further in a conic, having two points in common with r: each of these points, in the first case, or one of them, in the second case, is fixed, and is therefore a double point of F.

2. The problem of finding the limit of the 27 lines of a non-singular cubic surface F, when this tends to a surface F_0 consisting of three independent planes π_1, π_2, π_3, seems at first sight to be not quite determinate. We can, however, solve it by supposing that the lines $p_1 = \pi_2\pi_3$, $p_2 = \pi_3\pi_1$, $p_3 = \pi_1\pi_2$ intersect F in triads of points having well-defined limits $P_{11}P_{12}P_{13}$, $P_{21}P_{22}P_{23}$, $P_{31}P_{32}P_{33}$, the nine points $P_{\alpha\beta}$ and the point $O = \pi_1\pi_2\pi_3$ being distinct.

Let us notice, for this purpose, that the envelope of F has then as limit the net of planes through O counted three times, plus the 9 nets of planes of centres $P_{\alpha\beta}$ each counted once.† Since the 27 lines of F are the axes of as many pencils of bitangent planes of this surface (§ 1), it

† This is a very particular case of a result recently established by B. Segre, 'On limits of algebraic varieties, in particular of their intersections and tangential forms', *Proc. London Math. Soc.* (II), vol. 47 (1941), pp. 351–403, §§ 37, 38, where further references are given.

follows that each line r_0 which is limit of a line r of F must be the axis of a pencil having at least the multiplicity 2 for the degenerating envelope, so that either r_0 goes through O or it joins two points $P_{\alpha_1\beta}$, $P_{\alpha_2\gamma}$.†

We now observe that the first alternative has to be excluded. In fact, one (at least) of the three planes π_1, π_2, π_3, say π, does not contain r_0 and therefore not even r; the point πr_0, which is the limit of πr as $F \to F_0$, necessarily belongs to the limit of FF_0: so that it cannot coincide with O, since otherwise this point would coincide with one of the points $P_{\alpha\beta}$.

We remark next that a line $P_{\alpha_1\beta}P_{\alpha_2\gamma}$ cannot be the limit of two distinct lines of F. The limiting envelope has indeed this for double line, a generic plane through it having two distinct points of contact (namely, $P_{\alpha_1\beta}$ and $P_{\alpha_2\gamma}$); while a line limit of two distinct double lines of a variable envelope, if still double for the limiting envelope, is such that a generic plane through it must have two coincident points of contact with this limiting envelope.‡

We thus obtain the following result:

When F tends to F_0, the 27 lines of F have as limits the 27 distinct lines $P_{\alpha_1\beta}P_{\alpha_2\gamma}$ (with α_1, α_2, β, $\gamma = 1, 2, 3$, $\alpha_1 \neq \alpha_2$).

We add that:

If a surface F tends to a limiting position F_0 in such a way that two skew lines r, r' have as limits two incident lines r_0, r'_0, then the point $A_0 = r_0 r'_0$ is (at least) double for F_0.

Let us choose generically a point B, and consider the line s through B which intersects r and r', in A and A' say; then we have

$$\lim_{F\to F_0} A = A_0, \qquad \lim_{F\to F_0} A' = A_0, \qquad \lim_{F\to F_0} s = A_0 B,$$

so that at least two of the intersections of F_0 with $A_0 B$ (which is a generic line through A_0) are absorbed by the point A_0.

3. From §§ 1, 2 it follows that a non-singular cubic surface F, varying continuously, has always 27 distinct lines, whose corresponding continuous variation *preserves their mutual incidence relations.*

If F belongs to a continuous system, in correspondence with the different circulations of F within the system, we consequently obtain a *group of substitutions upon the* 27 *lines of* F, preserving the incidence relations of these lines.

The totality of the complex non-singular cubic surfaces of ordinary

† Throughout this chapter we indicate by a Greek letter α, β, or γ (possibly with an index) any one of the numbers 1, 2, 3, the same letter with different indices representing different numbers.

‡ This result appears almost as evident in its dual form.

space is *connected*. In fact, if we consider—within the complex projective space S (of complex dimension 19 and real dimension 38) representative of all the cubic surfaces of ordinary space—the algebraic variety V (of real dimension 36 and order 32) which represents the singular surfaces, we see that the open topological manifold

$$\Sigma = S - V$$

(of real dimension 38) is connected and represents the above-mentioned totality. We shall establish later that the group of substitutions defined by this totality is the group $\mathfrak{G}$ of *all* the substitutions among the 27 lines which preserve their incidence relations. We shall, moreover, prove (§ 24) that the real points of Σ constitute five distinct connected manifolds Σ_i (each of dimension 19), and obtain (section IX) the corresponding groups Γ_i ($i = 1, 2,..., 5$), which are, of course, subgroups of $\mathfrak{G}$.

If any non-singular cubic surface F is given, we may subject it to a continuous variation and let it tend (as in § 2) to a surface F_0 broken up into three planes, in such a way that each intermediate position is non-singular. This fact is quite obvious if we remain in the complex field, as we do in the present and following chapter; but we shall demonstrate later its validity in the real field also. The continuous variation from F to F_0 induces, then, a well-defined *one-to-one correspondence between the* 27 *lines of F and the* 27 *lines $P_{\alpha_1\beta} P_{\alpha_2\gamma}$ considered in* § 2. By arbitrarily changing the manner of degeneration of F into F_0 this correspondence can only be altered by the substitutions of a group, which clearly is $\mathfrak{G}$ if we consider the matter in the complex field, and Γ_i if the image point of F in S belongs to Σ_i and we remain in the real field.

II. A new notation and a graphical representation for the 27 lines

4. We can now introduce a new notation, and also give a very simple graphical (plane) representation, for the 27 lines of any non-singular cubic surface F.

We denote the line of F which (§ 3) corresponds to $P_{\alpha_1\beta} P_{\alpha_2\gamma}$ by a symbol, such as *abc*, formed by three figures, one of which is zero, while the other two are equal to β and γ and have respectively the α_1th and α_2th place.† The 27 lines of F are therefore

$$\begin{array}{ccccccccc} 011 & 012 & 013 & 101 & 201 & 301 & 110 & 120 & 130 \\ 021 & 022 & 023 & 102 & 202 & 302 & 210 & 220 & 230 \\ 031 & 032 & 033 & 103 & 203 & 303 & 310 & 320 & 330. \end{array} \qquad \text{(I)}$$

† Another, but less convenient (even if apparently more symmetrical), representation could be obtained by associating with this line the symbol $\alpha\beta\gamma$, where $\alpha\, \alpha_1\, \alpha_2$ is supposed to be an even permutation of the numbers 1, 2, 3.

A suggestive graphical representation is given, as in Fig. 1, by associating with each line of F the projection on to a plane (from a point not belonging to F_0) of the corresponding line $P_{\alpha_1\beta}P_{\alpha_2\gamma}$. We indicate by 1, 2, 3 the three lines of a pencil (dotted in Fig. 2) which are respectively the projections of p_1, p_2, p_3, and by $1_1\,1_2\,1_3$, $2_1\,2_2\,2_3$, $3_1\,3_2\,3_3$

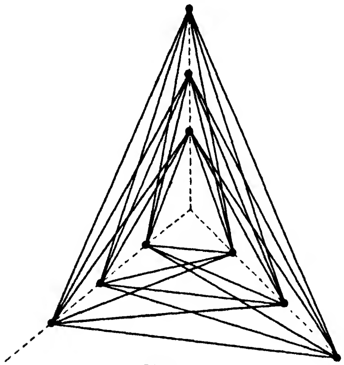


FIG. 1

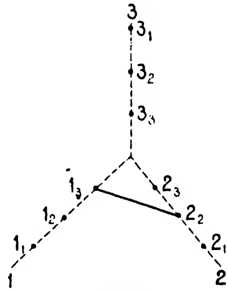


FIG. 2

their triads of points which are projections of $P_{11}P_{12}P_{13}$, $P_{21}P_{22}P_{23}$, $P_{31}P_{32}P_{33}$: so that, for instance, the line $1_3\,2_2$ of Fig. 2 is the graphical representation of the line of F which we have symbolically denoted by 320; there is, of course, no restriction in assuming those elements arranged as in Figs. 1, 2, which we shall always do in the future.

5. The incidence relations among the 27 lines are expressed by the following theorem.

Two lines abc and a′b′c′ are incident if, and only if, one and only one of the equalities $a = a'$, $b = b'$, $c = c'$ holds.

Two lines of the type $\alpha_1\beta_1 0$, $\alpha_2\beta_2 0$ (with $\alpha_1 \neq \alpha_2$, $\beta_1 \neq \beta_2$) are, in fact, incident by virtue of the final remark of § 2, since they correspond to the lines $P_{1\alpha_1}P_{2\beta_1}$ and $P_{1\alpha_2}P_{2\beta_2}$ of the limiting surface F_0, which intersect in a *simple* point of this surface. It follows that two lines such as $\alpha_1\beta_1 0$ and $\alpha_1\beta_2 0$ are skew, since they are both incident to two distinct lines of F, for instance $\alpha_2\beta_3 0$ and $\alpha_3\beta_3 0$. Moreover, two lines of F such as $\alpha_1\beta\, 0$ and $\alpha_2 0\gamma$ (with $\alpha_1 \neq \alpha_2$) are certainly skew, this being true of their corresponding lines on F_0 ($P_{1\alpha_1}P_{2\beta}$ and $P_{1\alpha_2}P_{3\gamma}$); and, finally, two lines such as $\alpha\beta 0$ and $\alpha 0\gamma$ are incident, since the latter is skew with $\alpha_1\beta_1 0$ and $\alpha_2\beta_2 0$, which belong to a plane through the former.

The result just proved is equivalent to the following:

The condition of incidence of two lines of F is that their images on F_0

either belong to the same plane π_α or contain the same point $P_{\alpha\beta}$, these two circumstances, however, not occurring at the same time.

Consequently we are now in a position to see whether two lines of F, given by their graphical representations, are incident or skew. In Fig. 3, for instance, we have represented one of the 27 lines (the line

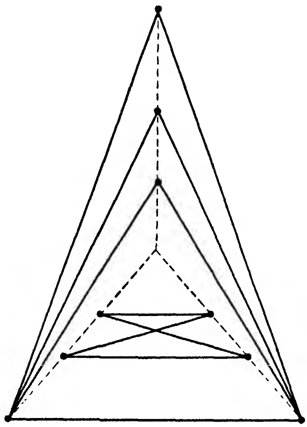


FIG. 3

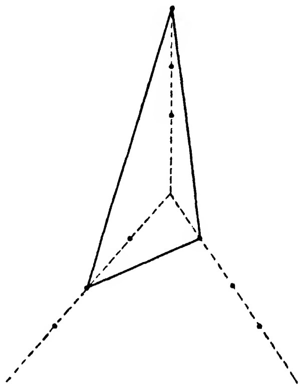


FIG. 4

110), together with the remaining lines of F incident to it: these are *10 in number*, and can clearly be distributed in 5 pairs of intersecting lines.

III. Application to certain configurations formed by some of the 27 lines

6. The notation and the graphical representation given in the last section for the 27 lines of a non-singular cubic surface offer intuitive and helpful means for the study of the relations between those lines. Postponing group-theoretical considerations to another chapter, we now confine ourselves to the investigation of certain configurations formed by some of the 27 lines.

For instance, we see at once from § 5 that F has 45 *tritangent planes* (that is, planes containing 3 of its lines): 27 of them are joins of 3 lines of the type $0\beta\gamma$, $\alpha 0\gamma$, $\alpha\beta 0$, and can be represented symbolically by $(\alpha\beta\gamma)$ or graphically as in Fig. 4; the 18 which remain are joins of 3 lines of the type $\alpha_1\beta_1 0$, $\alpha_2\beta_2 0$, $\alpha_3\beta_3 0$, and can be represented symbolically by the square matrices

$$\begin{pmatrix} \alpha_1 & \beta_1 & 0 \\ \alpha_2 & \beta_2 & 0 \\ \alpha_3 & \beta_3 & 0 \end{pmatrix}$$

(the rows of which can be freely interchanged), or graphically as in Fig. 5.

A first configuration connected with the 27 lines and 45 tritangent planes of F arises as follows. On any line r of F the conics which are the residual intersections of F with the planes through it determine

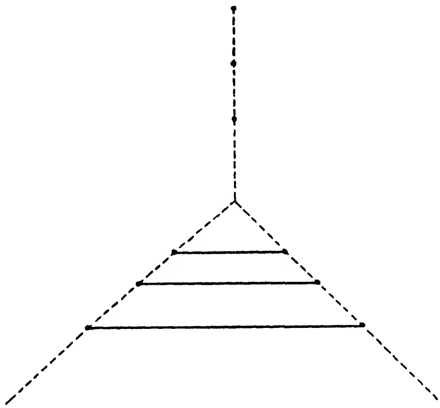


FIG. 5

pairs of an *involution*, the double points of which are called the *parabolic points* of r; the plane touching F at such a point intersects F along r and a conic touching r at it, and is called a *parabolic plane*. We see that:

The 3 *pairs of parabolic points of* 3 *coplanar non-concurrent lines of* F *are the* 3 *pairs of opposite vertices of a plane quadrilateral.*

In fact, if a plane ρ meets F along 3 lines r_1, r_2, r_3, intersecting two by two in 3 distinct points $A_1 = r_2 r_3$, $A_2 = r_3 r_1$, $A_3 = r_1 r_2$, and if R_i is a parabolic point of r_i ($i = 1, 2, 3$), then there is a parabolic plane ρ_i which intersects F along r_i and a conic touching this line at R_i. It follows that the pencil determined by F and $\rho_1 \rho_2 \rho_3$ contains a surface broken up into ρ and a quadric touching r_1, r_2, r_3 at R_1, R_2, R_3 respectively; so that, according as this quadric is singular or non-singular, the points R_1 and R_2 are collinear with R_3 or with the harmonic conjugate of R_3 with respect to A_1, A_2, which is the further parabolic point of r_3.

If 3 coplanar lines of F have a point in common, this is called an *Eckardt point* of F and is one of the parabolic points of each of those 3 lines. In this case the same argument proves that *the other* 3 *parabolic points are in a line.*

We can say that:

The 54 *parabolic points of the* 27 *lines of a cubic surface F without Eckardt points are situated* 3 *by* 3 *upon* 180 *lines, distributed* 4 *by* 4 *on the* 45 *tritangent planes of F.*

7. *The maximum number of non-intersecting lines of F is* 6. Also it is easily seen that the 27 lines can be distributed in sets of 2, 3, 4, 5, 6 non-intersecting lines, as follows:

216 *doublets,*	of which	54	have an image of the type represented in Fig. 6
		162	” ” ” ” ” 7
720 *triplets,*	of which	18	” ” ” ” ” 8
		324	” ” ” ” ” 9
		162	” ” ” ” ” 10
		216	” ” ” ” ” 11
1080 *quadruplets,*	of which	108	” ” ” ” ” 12
		162	” ” ” ” ” 13
		162	” ” ” ” ” 14
		648	” ” ” ” ” 15
432 *quintuplets of the* 1*st kind,*	of which	108	” ” ” ” ” 16
		324	” ” ” ” ” 17
216 *quintuplets of the* 2*nd kind,*	of which	54	” ” ” ” ” 18
		162	” ” ” ” ” 19
72 *sextuplets,*	of which	18	” ” ” ” ” 20
		54	” ” ” ” ” 21

Each triplet has a well-defined complementary triplet, constituted by lines incident with each of its 3 lines. *Two complementary triplets are therefore the complete intersection of F with a quadric*; both of them have graphical representations of the same type, as is shown by Figs. 8–11, where one of the two conjugate triplets is marked by dotted lines.

Quintuplets are of 1st or of 2nd kind, according as they have one or two common transversals; these common transversals are shown in Figs. 16, 17 and 18, 19 by dotted lines.

Each sextuplet has a well-defined complementary sextuplet (marked with dotted lines in Figs. 20, 21); two complementary sextuplets make up a so-called *double-six,* and are related by a one-to-one correspondence in such a way that two lines correspond if, and only if, they are skew. *There is one and only one double-six containing two given skew lines of F as corresponding lines*: it consists of these two lines and the 10 lines of *F* incident to one and only one of them.

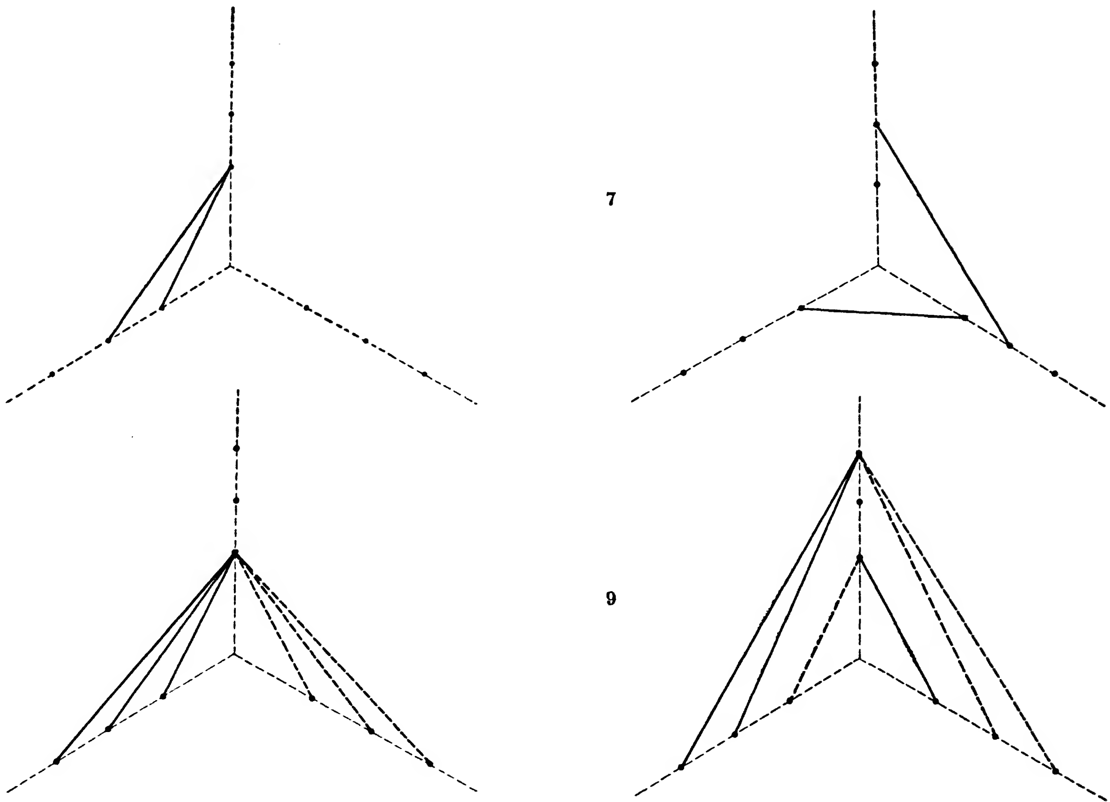
There are in all 36 *double-sixes.* Nine of these have an image of the

6

7

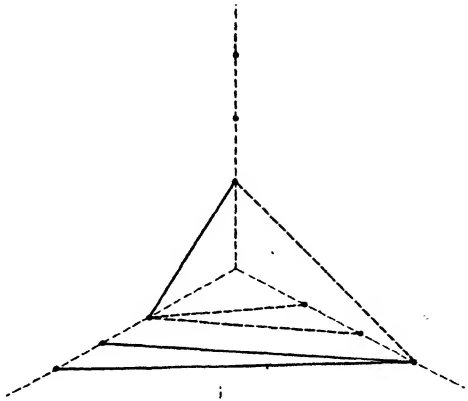
8

9

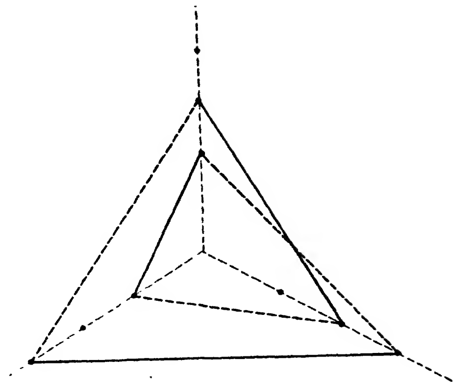


FIGS. 6–9

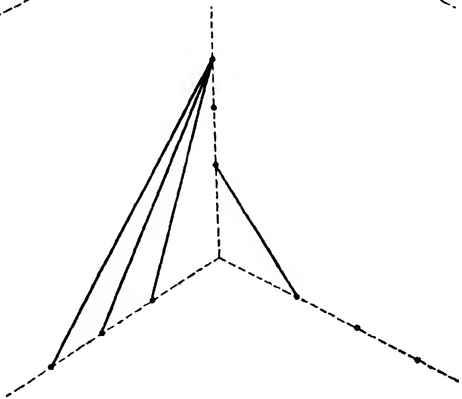
10



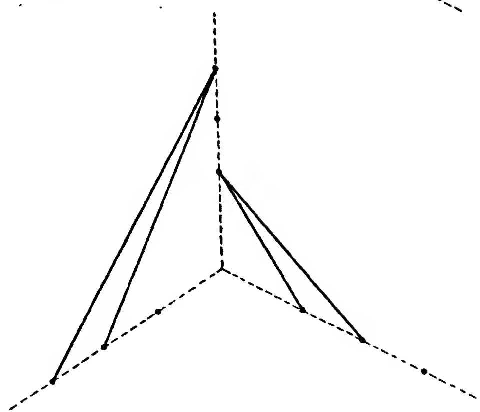
11



12

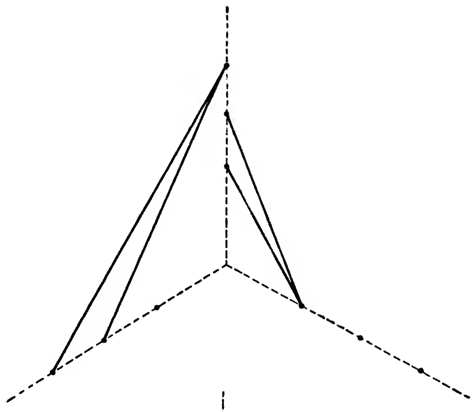


13

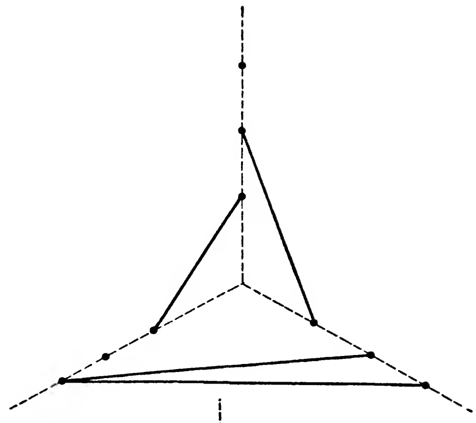


FIGS. 10–13

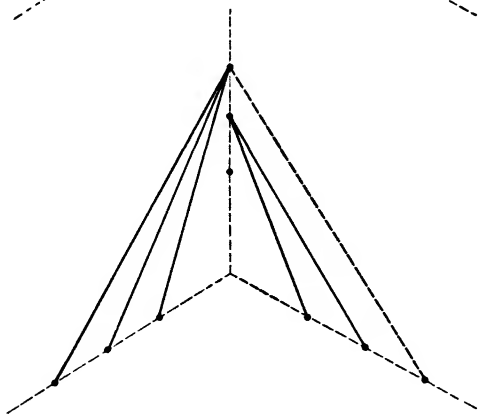
14



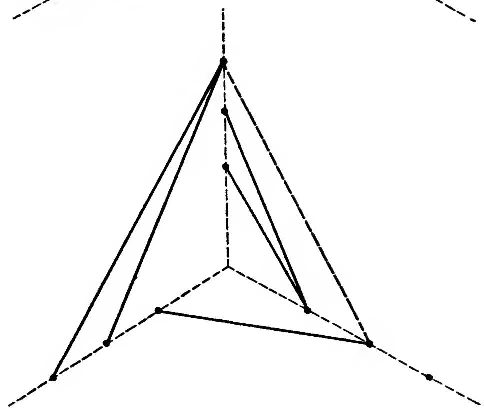
15



16

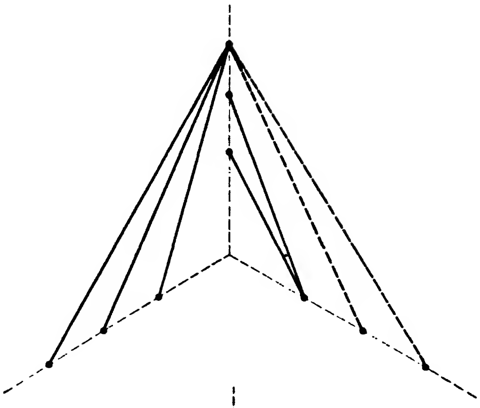


17

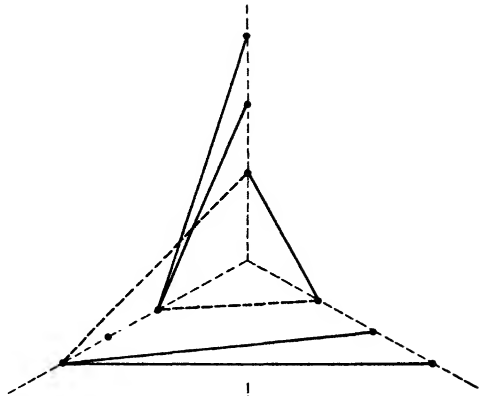


Figs. 14–17

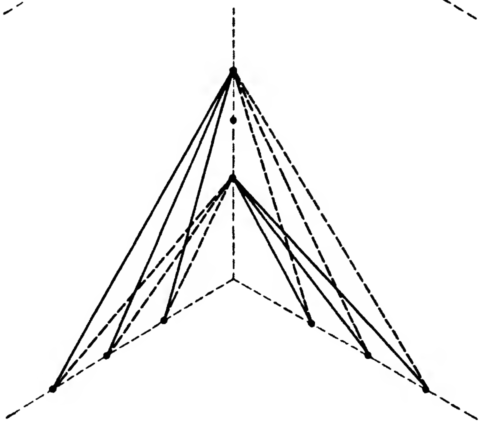
18



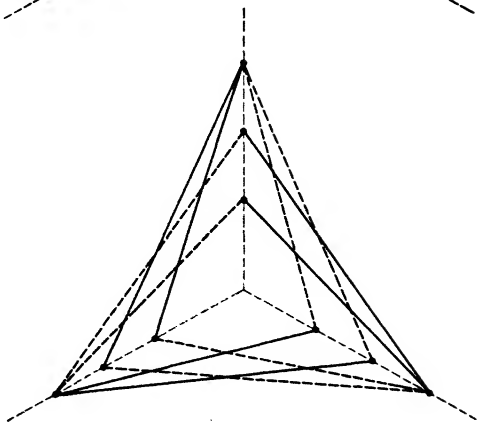
19



20



21



FIGS. 18–21

type represented in Fig. 20, which consists of the 12 lines through two points α_{β_1}, α_{β_2} of a triad: if α_β is the third point of the triad, we shall denote the corresponding double-six by the symbol $\{\alpha\beta\}$, and represent it graphically by a small circle round the point α_β (Fig. 22). The remaining 27 double-sixes have an image of the type marked in Fig. 21, which consists of the 12 lines through three points 1_α, 2_β, 3_γ of different

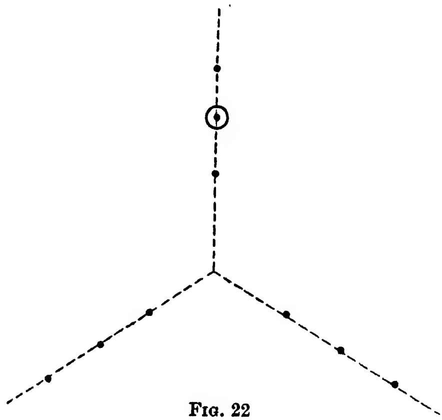
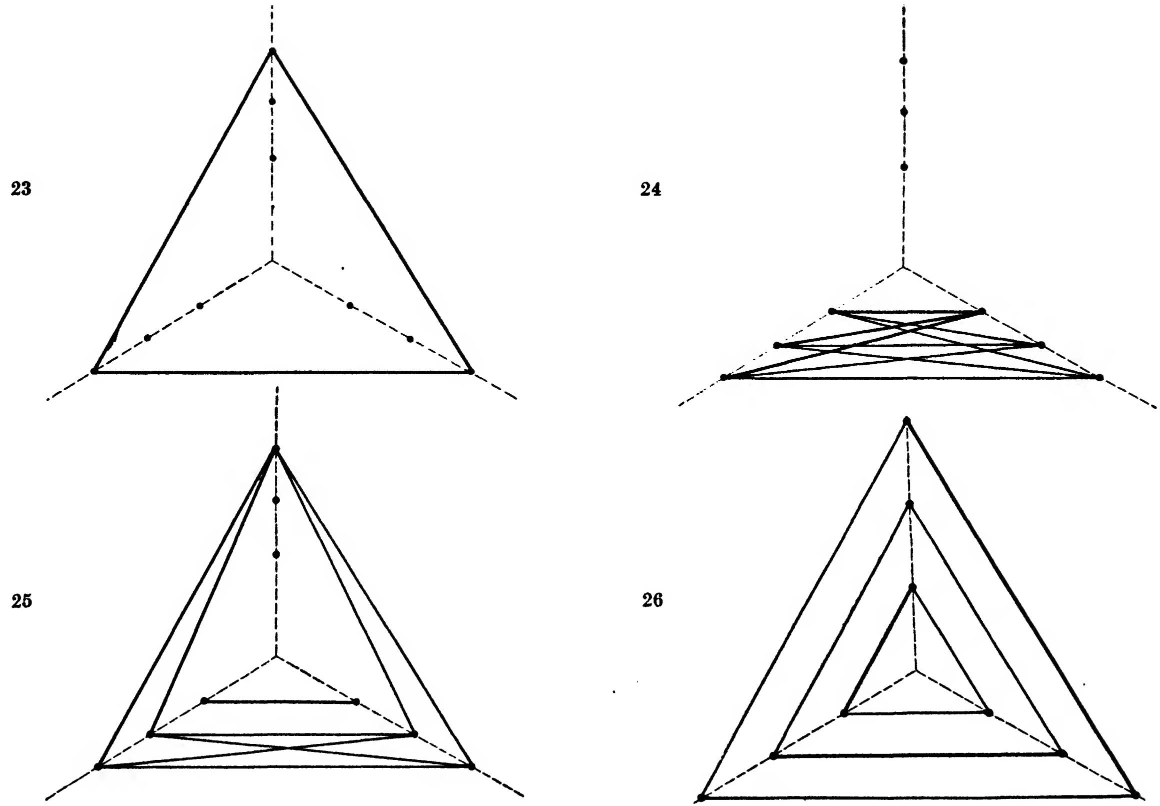


FIG. 22

triads, with the exception of the sides of the triangle determined by these points: we shall denote the corresponding double-six by the symbol $\{\alpha\beta\gamma\}$, and represent it graphically by the (heavily printed) sides of the triangle $1_\alpha 2_\beta 3_\gamma$ (Fig. 23).

8. If we start from any of the 720 triplets (§ 7), we have exactly 6 lines of F which are incident to two of its lines and skew to the remaining one; 6 such lines can be split up (uniquely) into two triplets: each of the 3 triplets thus obtained is related in the same way to the other two, and we say that the 3 triplets are *associated* and that their 9 lines constitute a *Steiner set*. All this follows almost at once from the graphical representation; according as the triplet initially considered is of the type represented by Fig. 8, by Figs. 9 or 10, or by Fig. 11, the graphical representation of the corresponding Steiner set is said to be of the 1st, 2nd, or 3rd type, respectively given by Figs. 24, 25, or 26 (where one of the triplets is marked with a thicker stroke). We see at once, moreover, that the 9 lines of a Steiner set can be split up into 3 associated triplets in two ways, the complementary triplets forming two other sets of 3 associated triplets; we thus obtain in all



Figs. 23–26

3 Steiner sets, including all the 27 lines of F, and two by two related in the same manner: we call them 3 *complementary Steiner sets.*

With the 720 triplets we can form $720:3 = 240$ triads of associated triplets. The Steiner sets are consequently $240:2 = 120$ in number, there being 3 of the 1st type, 81 of the 2nd type, and 36 of the 3rd type; for the triads of complementary Steiner sets these numbers must be divided by 3.

The 9 lines of a Steiner set can be written as the elements of a third order determinant, whose rows and columns give the two triads of mutually associated triplets that can be formed with them; two elements of such a determinant are skew or incident, according as they do or do not belong to the same row or column. Each term in the expansion of the determinant gives 3 lines of a tritangent plane; the 3 positive and the 3 negative terms thus give rise to a so-called *Steiner trihedral pair*, namely, a pair of trihedra intersecting in 9 lines belonging to F (which then constitute a Steiner set). The possibility of combining 3 Steiner sets so as to give all the 27 lines is well known; here we see its very simple relation with our notation for the 27 lines, since one triad of complementary Steiner sets (that of the 1st type) is represented by the 3 determinants of the table (I) of § 4. The other 27+12 triads (of the 2nd and 3rd type respectively) are likewise given by the tables

$$\begin{matrix} 0\beta_1\gamma_1 & \alpha_1 0\gamma_3 & \alpha_1 0\gamma_2 \\ \alpha_1\beta_3 0 & 0\beta_2\gamma_2 & 0\beta_2\gamma_3 \\ \alpha_1\beta_2 0 & 0\beta_3\gamma_2 & 0\beta_3\gamma_3 \end{matrix} \qquad \begin{matrix} \alpha_1 0\gamma_1 & \alpha_3\beta_1 0 & \alpha_2\beta_1 0 \\ 0\beta_1\gamma_3 & \alpha_2 0\gamma_2 & \alpha_3 0\gamma_2 \\ 0\beta_1\gamma_2 & \alpha_2 0\gamma_3 & \alpha_3 0\gamma_3 \end{matrix} \qquad \begin{matrix} \alpha_1\beta_1 0 & 0\beta_3\gamma_1 & 0\beta_2\gamma_1 \\ \alpha_3 0\gamma_1 & \alpha_2\beta_2 0 & \alpha_2\beta_3 0 \\ \alpha_2 0\gamma_1 & \alpha_3\beta_2 0 & \alpha_3\beta_3 0 \end{matrix} \qquad \text{(II)}$$

and

$$\begin{matrix} 0\beta_1\gamma_1 & \alpha_1 0\gamma_3 & \alpha_3\beta_2 0 \\ \alpha_1\beta_3 0 & 0\beta_2\gamma_2 & \alpha_2 0\gamma_1 \\ \alpha_3 0\gamma_2 & \alpha_2\beta_1 0 & 0\beta_3\gamma_3 \end{matrix} \qquad \begin{matrix} \alpha_1 0\gamma_1 & \alpha_3\beta_1 0 & 0\beta_3\gamma_2 \\ 0\beta_1\gamma_3 & \alpha_2 0\gamma_2 & \alpha_1\beta_2 0 \\ \alpha_2\beta_3 0 & 0\beta_2\gamma_1 & \alpha_3 0\gamma_3 \end{matrix} \qquad \begin{matrix} \alpha_1\beta_1 0 & 0\beta_3\gamma_1 & \alpha_2 0\gamma_3 \\ \alpha_3 0\gamma_1 & \alpha_2\beta_2 0 & 0\beta_1\gamma_2 \\ 0\beta_2\gamma_3 & \alpha_1 0\gamma_2 & \alpha_3\beta_3 0, \end{matrix} \qquad \text{(III)}$$

where $\alpha_1\alpha_2\alpha_3$, $\beta_1\beta_2\beta_3$, $\gamma_1\gamma_2\gamma_3$ are any three permutations of the numbers 1, 2, 3.

From the graphical representation it follows immediately that each of the 120 Steiner sets can also be defined, in 3 different ways, as the set of the residual intersections of F with the planes joining 2 by 2 (in all possible manners) the 6 lines of one of the 360 pairs of complementary triplets. The sum of two such sets of 9 and 6 lines is obtainable in this manner in 10 different ways, and is simply one of the 36 *sets of* 15 *lines residual to a double-six*, whose incidence relations are therefore characteristic.

9. Each of the 27 lines of F being incident to 10 others (§ 5),† we get on F a set of $\frac{1}{2}.27.10 = 135$ points—which we call *k-points*—each being the intersection of two of the 27 lines; and we say that a k-point and a line of F are *associated*, when the 2 lines intersecting in the former are coplanar with the latter. A k-point has a well-defined associated line, but every line has 5 associated k-points (§ 5), constituting what we call a *set of* 5 *associated k-points.*

Each of the 45 tritangent planes contains three lines of F, and is intersected along a line different from these by 32 other tritangent planes; hence we have $\frac{1}{2}.45.32 = 720$ lines not lying on F—which we call *k-lines*—each of which is the intersection of two tritangent planes, and therefore meets F in 3 k-points; the 720 k-lines are clearly the edges of the 120 Steiner trihedral pairs (§ 8). If we call an edge and the opposite face of a Steiner trihedron *associated*, we see that each k-line has a well-defined associated tritangent plane, and each tritangent plane has $\frac{1}{2}.32 = 16$ associated k-lines.

The 3 lines of F associated with the 3 k-points of a k-line are in a plane, which is the tritangent plane associated with this k-line. It follows that two k-points are upon a k-line (so that their join intersects F further in a k-point) if, and only if, their associated lines are incident; and that two k-lines intersect in a k-point if, and only if, their associated tritangent planes intersect in a line of F (associated to the k-point). Hence we easily deduce that:

The 3 *lines of a tritangent plane constitute a triangle, each side of which is associated with the opposite vertex; these* 3 *sides have as further associates* 3 *sets of* 4 *k-points, which are the vertices of* 3 *desmic tetrahedra: that is, any two points of different sets are collinear with a point of the remaining set. The* 16 *lines containing* 2 (*and consequently* 3) *points of different sets are the k-lines associated to the tritangent plane with which we began.*

This result can be immediately verified by reference to the limiting configuration considered in section I. Starting, for instance, from the tritangent plane (333) we get the 3 tetrahedra

$$\begin{array}{llll} P_{11}, & P_{12}, & P_{21}P_{31}.P_{22}P_{32}, & P_{21}P_{32}.P_{22}P_{31}, \\ P_{21}, & P_{22}, & P_{31}P_{11}.P_{32}P_{12}, & P_{31}P_{12}.P_{32}P_{11}, \\ P_{31}, & P_{32}, & P_{11}P_{21}.P_{12}P_{22}, & P_{11}P_{22}.P_{12}P_{21}, \end{array}$$

† Here we suppose the 10 points of incidence to be always distinct, that is, we exclude the case in which F has 3 (necessarily coplanar) lines with a point in common; but it is easy to see the modifications required if this is not the case, that is (§ 6), if F has Eckardt points.

which are obviously desmic. We call a tetrahedron of the type considered in our last theorem a *k-tetrahedron.*

10. Let us consider two tritangent planes, ω, ω' say, intersecting in a k-line. The 6 lines in which they meet the surface F belong to a well-defined Steiner set of 9 lines; and from the remaining $27-9=18$ lines it is possible—in three different ways—to obtain a double-six by suppressing 6 of them. Such a set of 6 lines, taken together with the 6 lines of F initially considered, gives a set of 12 lines, which we call a *12-set of lines.*

A 12-set of lines has the characteristic property of being—in two different ways—the complete intersection of F with the faces of a tetrahedron. The two tetrahedra intersecting F along the same 12-set of lines determine a pencil of quartic surfaces, one of which breaks up into F and a residual plane, which we call a *k-plane*; it follows that the 2 tetrahedra intersect this k-plane along the same quadrilateral—i.e. they are in perspective—the planes ω and ω' initially considered being 2 corresponding faces, and that F contains the 6 vertices of this quadrilateral.†

The total number of 12-sets is therefore $\frac{1}{4}.720.3 = 540$, since a single 12-set can be obtained in the indicated manner in 4 different ways, and is equal to the number of the k-planes. Moreover, each side of the above-considered quadrilateral is a k-line, and each k-line belongs to 3 k-planes. The number of 12-sets can also be obtained by noticing that each 12-set can be derived from one of the 36 sets of 15 lines residual to a double-six (already considered at the end of § 8), by suppressing one of its 15 triads of coplanar lines: we have, in fact, $36.15 = 540$; from this remark and our earlier properties of the 12-sets, it follows at once that:

The 15 *lines of F residual to a double-six can always be considered—in* 6 *different ways—as the complete intersection of F with* 5 *of its tritangent planes; the* 15 *planes which contain* 3 *of these* 15 *lines can be*

† If $\lambda_i = 0$, $\lambda_i' = 0$ $(i = 1, 2, 3, 4)$ are the equations of the corresponding faces of the two tetrahedra and $\mu = 0$ is the equation of the k-plane, this plane and F constitute a surface of the pencil determined by $\lambda_1\lambda_2\lambda_3\lambda_4 = 0$ and $\lambda_1'\lambda_2'\lambda_3'\lambda_4' = 0$. Since the planes $\lambda_i = 0$, $\lambda_i' = 0$, $\mu = 0$ are in a pencil, we can without restriction suppose $\lambda_i' \equiv \lambda_i+\mu$, and we obtain for F the equation

$$\mu^3+\sigma_1\mu^2+\sigma_2\mu+\sigma_3 = 0,$$

where σ_r $(r = 1, 2, 3)$ is the sum of the products of the λ_i's taken r by r; the cubic surface F contains the 6 vertices $\lambda_i = \lambda_j = \mu = 0$ of the quadrilateral considered above, and the 12 lines $\lambda_i = \lambda_j+\mu = 0$ $(i, j = 1, 2, 3, 4;\ i \neq j)$ constitute on it a 12-set.

distributed in 6 *systems of* 5 *planes containing all the* 15 *lines, so that each plane belongs to* 2 *different systems and every* 2 *systems have one plane in common.*

All the foregoing results can immediately be verified graphically. We

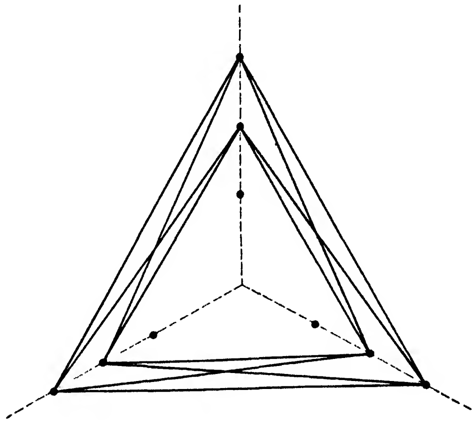


FIG. 27

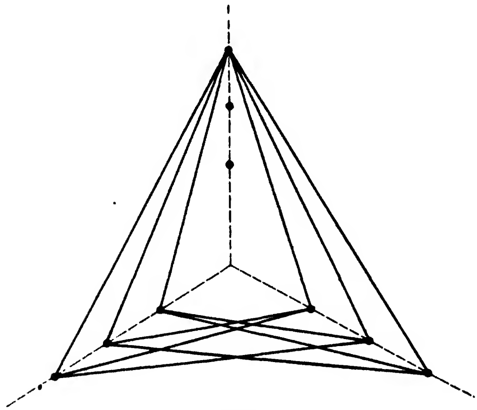


FIG. 28

obtain for the 540 12-sets representations of the five different types given in Figs. 27–31, which account respectively for 27, 54, 81, 162, and 216 of them. In the notation of § 6, the corresponding types of perspective tetrahedra are indicated in the following five tables, where homologous planes are written in the same column; on the left are represented the planes which, together with the faces of either of such

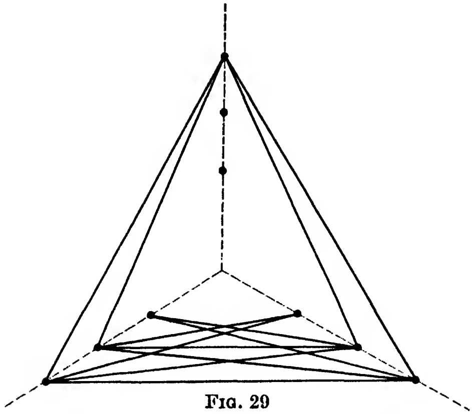


Fig. 29

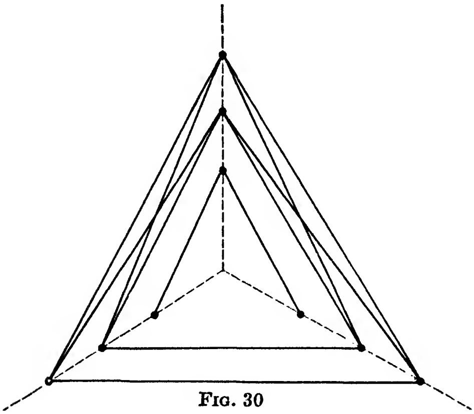


Fig. 30

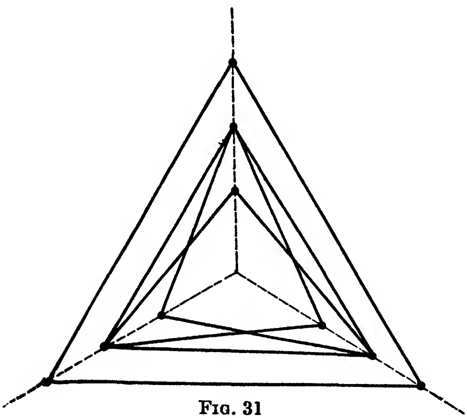


Fig. 31

tetrahedra constitute a system of 5 tritangent planes of the type defined previously.

$$(333)\quad\left\{\begin{array}{cccc}(111) & (122) & (212) & (221)\\ (222) & (211) & (121) & (112)\end{array}\right.$$

$$\begin{pmatrix}110\\220\\330\end{pmatrix}\quad\left\{\begin{array}{cccc}\begin{pmatrix}230\\310\\120\end{pmatrix} & (321) & (131) & (211)\\ \begin{pmatrix}320\\130\\210\end{pmatrix} & (231) & (311) & (121)\end{array}\right.$$

$$(331)\quad\left\{\begin{array}{cccc}(111) & (221) & \begin{pmatrix}130\\210\\320\end{pmatrix} & \begin{pmatrix}120\\230\\310\end{pmatrix}\\ \begin{pmatrix}130\\220\\310\end{pmatrix} & \begin{pmatrix}110\\230\\320\end{pmatrix} & (121) & (211)\end{array}\right.$$

$$\begin{pmatrix}120\\210\\330\end{pmatrix}\quad\left\{\begin{array}{cccc}(111) & (222) & \begin{pmatrix}012\\021\\033\end{pmatrix} & \begin{pmatrix}102\\201\\303\end{pmatrix}\\ (221) & (112) & \begin{pmatrix}101\\202\\303\end{pmatrix} & \begin{pmatrix}011\\022\\033\end{pmatrix}\end{array}\right.$$

$$(333)\quad\left\{\begin{array}{cccc}(111) & (322) & (232) & (223)\\ (222) & \begin{pmatrix}011\\023\\032\end{pmatrix} & \begin{pmatrix}101\\203\\302\end{pmatrix} & \begin{pmatrix}110\\230\\320\end{pmatrix}\end{array}\right..$$

We can say in conclusion, taking into account § 9, that:

The 135 *k-points,* 720 *k-lines, and* 540 *k-planes form a configuration, in which each point belongs to* 16 *lines and* 24 *planes; each line contains* 3 *points and belongs to* 3 *planes (the* 720 *k-lines intersecting, moreover,* 3 *by* 3 *in* 240 *points not belonging to the configuration); each plane contains* 6 *points and* 4 *lines, the former being the vertices of the quadrilateral formed by the latter. A given k-point belongs to one set of* 5 *associated k-points, from which it is possible to form* 4 *tetrahedra having the given k-point as a vertex; each of these tetrahedra is desmic with two other k-tetrahedra: and the vertices and sides of the* 4 *pairs of tetrahedra so defined—projected from the k-point initially considered—give, respectively, the* $4.4 = 16$ *k-lines and the* $4.6 = 24$ *k-planes going through that point.*

By reversing one of the previous arguments, we have, moreover, that:

The faces of two perspective tetrahedra intersect the fundamental plane

of the homology along the same quadrilateral, and have as further mutual intersection 12 *lines belonging to a (possibly singular) cubic surface; such a surface goes through the vertices of the quadrilateral and contains these* 12 *lines as a* 12-*set.*

11. We have immediately, from our representation, that 2 distinct double-sixes have in common either 4 lines (forming 2 pairs of incident lines without points in common, so that all the 4 lines meet a fifth one) or 6 lines (forming 2 complementary triplets); and we call them, respectively, mutually *permutable* or *non-permutable*.† We see at once, moreover, that the $6+6 = 12$ lines of 2 non-permutable double-sixes which are not common to them both form a third double-six, non-permutable with each of them; the 3 double-sixes are in a symmetrical relation, and we say that they are *associated.*

Each of the 36 double-sixes is permutable with 15 others, and non-permutable with 20 others; so that we have in all $\frac{1}{2}.36.15 = 270$ pairs of permutable double-sixes, and $\frac{1}{6}.36.20 = 120$ triads of associated double-sixes. The former have graphical representations of 3 different types, given in Figs. 32–4 (where the 4 lines common to the two double-sixes are dotted), which account respectively for 27, 81, and 162 of them; the latter have also graphical representations of 3 different types, given in Figs. 35–7, which account respectively for 3, 81, and 36 of them:‡ they are in a strict relationship with the 120 Steiner sets, since a triad of associate double-sixes consists of 18 distinct lines determining residually the $27-18 = 9$ lines of such a set (the residual sets of the triads of double-sixes represented in Figs. 35, 36, 37 being respectively those represented in Figs. 24, 25, 26).

By virtue of § 8, the 120 triads of associated double-sixes can be distributed in 40 sets of 3 complementary triads; moreover, as is shown by the graphical representation:

There are exactly 6 *double-sixes permutable with* 3 *associate ones, and they can be arranged uniquely in* 2 *triads of associate double-sixes, making, with the triad initially considered, a set of* 3 *complementary triads.*

† We shall see later on (§ 13) the reason for these descriptions.

‡ In Fig. 36 a line is drawn with double stroke, in order to emphasize its belonging to two different triangles.

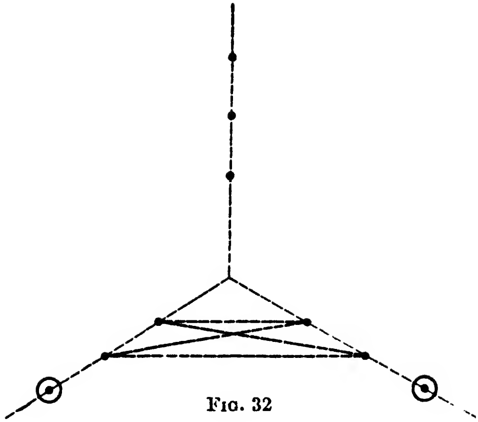


Fig. 32

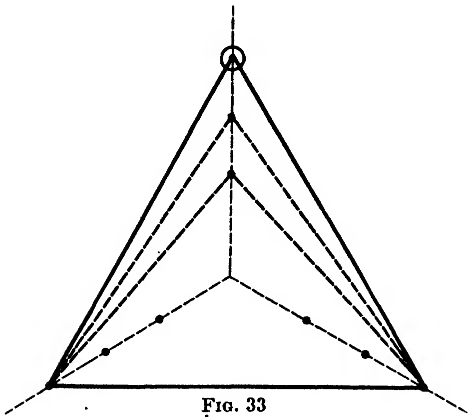


Fig. 33

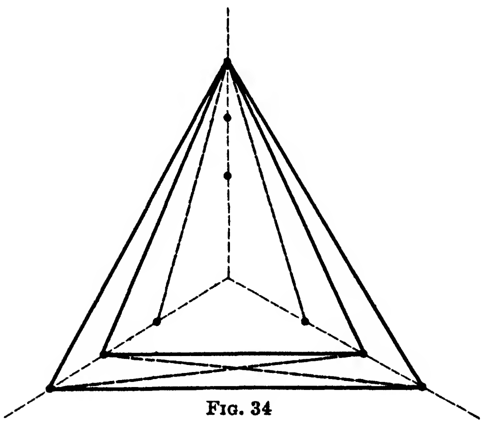


Fig. 34

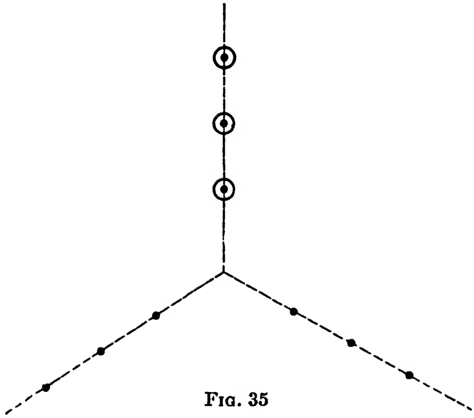


Fig. 35

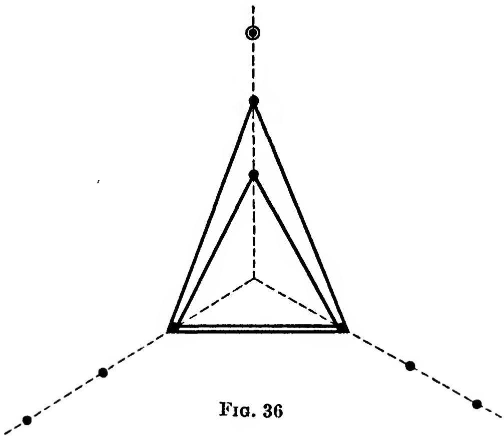


Fig. 36

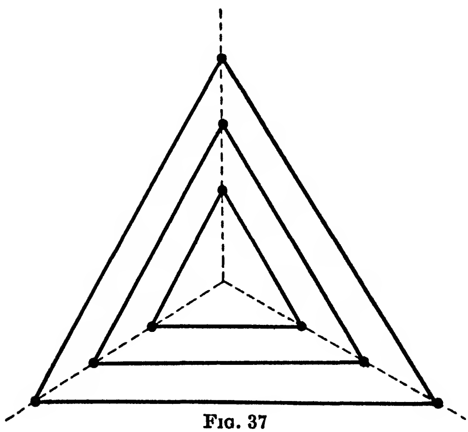


Fig. 37

II

THE GROUP $\mathfrak{G}$ OF THE 27 LINES

IV. The group $\mathfrak{G}$: its representation and some of its sub-groups

12. We now consider the group $\mathfrak{G}$ consisting of all the substitutions among the 27 lines which preserve their incidence relations; we shall see how useful our notation and graphical representation is in studying it.

Every substitution of $\mathfrak{G}$ transforms each doublet, triplet,..., sextuplet, etc., into a similar set of lines, so that it induces a substitution among the 40 triads of complementary Steiner sets (§ 8), another among the 36 double-sixes (§ 7), another among the 45 tritangent planes (§ 6); similarly we have other substitutions among the 135 k-points, the 720 k-lines (§ 9), the 540 k-planes (§ 10), etc. Corresponding to $\mathfrak{G}$ we therefore obtain many groups of substitutions (among 40, 36, 45, 135, 720, 540,... elements) having different geometrical significance: but it is not difficult to see that $\mathfrak{G}$ and all these groups are transitive, and that any two are simply isomorphic.

Thus, for instance, the group $\mathfrak{G}'$ induced by $\mathfrak{G}$ among the 40 triads of complementary Steiner sets is simply isomorphic with $\mathfrak{G}$, since the 40 Steiner sets which contain one of the 27 lines have no other line in common. $\mathfrak{G}'$ is, moreover, transitive, as we obtain a substitution of $\mathfrak{G}$ transforming the triad (I) of § 4 into the triad (II) or (III) of § 8, simply by associating with each element of the table (I) the element which has the same place in the table (II) or (III).

We can at once derive sub-groups of $\mathfrak{G}$ by considering the totality of operations of $\mathfrak{G}$ which transform into itself one of the configurations connected with the 27 lines. Let us consider, for instance, the sub-group Γ which leaves fixed one of the 40 triads of complementary Steiner sets; and we can suppose this to be (I). Then, taking into account § 5, we see that Γ arises from the substitutions among the 9 points $P_{\alpha\beta}$ which preserve their alignments, so that Γ is of order $(3!)^4 = 1{,}296$; Γ clearly contains a self-conjugate sub-group of index 6—formed by the substitutions of $\mathfrak{G}$ transforming into itself each of the 3 Steiner sets (I)—which is simply isomorphic with the direct product of 3 symmetric groups of degree 3, the quotient group of Γ and this sub-group being isomorphic with the symmetric group of degree 3. Every substitution

of Γ can be obtained once and only once by transforming the table (I) into the following one:

$$\begin{array}{ccc} 0\beta_1'\gamma_1' \ 0\beta_1'\gamma_2' \ 0\beta_1'\gamma_3' & \alpha_1'0\gamma_1' \ \alpha_2'0\gamma_1' \ \alpha_3'0\gamma_1' & \alpha_1'\beta_1'0 \ \alpha_1'\beta_2'0 \ \alpha_1'\beta_3'0 \\ 0\beta_2'\gamma_1' \ 0\beta_2'\gamma_2' \ 0\beta_2'\gamma_3' & \alpha_1'0\gamma_2' \ \alpha_2'0\gamma_2' \ \alpha_3'0\gamma_2' & \alpha_2'\beta_1'0 \ \alpha_2'\beta_2'0 \ \alpha_2'\beta_3'0 \\ 0\beta_3'\gamma_1' \ 0\beta_3'\gamma_2' \ 0\beta_3'\gamma_3' & \alpha_1'0\gamma_3' \ \alpha_2'0\gamma_3' \ \alpha_3'0\gamma_3' & \alpha_3'\beta_1'0 \ \alpha_3'\beta_2'0 \ \alpha_3'\beta_3'0 \end{array}$$

(where $\alpha_1'\alpha_2'\alpha_3'$, $\beta_1'\beta_2'\beta_3'$, $\gamma_1'\gamma_2'\gamma_3'$ are any 3 permutations of the numbers 1, 2, 3), or in one derived from this by altering in the same manner the order of the 3 figures of each element; this is equivalent to a combination of the following operations upon the 3 square matrices constituting the table (I):

(i) the performing of arbitrary substitutions upon the rows of the 1st, 2nd, and 3rd matrices, and of the same substitutions respectively upon the columns of the 3rd, 1st, and 2nd matrices (these operations generate the sub-group of Γ considered above);

(ii) the interchange of two matrices, accompanied by the simultaneous interchange in all matrices of rows and columns.

We can say in conclusion that:

The group $\mathfrak{G}$ contains a complete set of 40 *conjugate sub-groups, each of which has order* 1,296, *index* 40, *and the same structure as* Γ; *the order of $\mathfrak{G}$ is therefore* $1{,}296 . 40 = 51{,}840$. *Every transformation of $\mathfrak{G}$ can be obtained by associating with each element of one of the above-mentioned tables the element which has the same place in one of the tables* (I), (II), *or* (III).

13. The group $\mathfrak{G}$ contains a conjugate set of 36 involutory operations, which we call *σ-transformations*, each of which interchanges the conjugate lines of 2 complementary sextuplets (§ 7) and leaves unaltered the remaining $27-12 = 15$. We shall denote by $[\alpha\beta]$ or $[\alpha\beta\gamma]$ respectively the σ-transformation thus defined by a double-six $\{\alpha\beta\}$ or $\{\alpha\beta\gamma\}$, and represent the former graphically by the same diagram as the latter (§ 7). The σ-transformation $[\alpha\beta\gamma]$ is explicitly given by the involutory substitution

$$[\alpha\beta\gamma] = \begin{pmatrix} 0\beta\gamma_1 & 0\beta\gamma_2 & \alpha_1 0\gamma & \alpha_2 0\gamma & \alpha\beta_1 0 & \alpha\beta_2 0 \\ \alpha 0\gamma_2 & \alpha 0\gamma_1 & \alpha_2\beta 0 & \alpha_1\beta 0 & 0\beta_2\gamma & 0\beta_1\gamma \end{pmatrix},$$

and has a fairly simple graphical interpretation; still simpler is the graphical interpretation of $[\alpha\beta]$, which is clearly reflected in the interchanging of the points α_{β_1} and α_{β_2}.

We can easily see that two σ-transformations are *permutable* if, and only if, the same is true of their corresponding double-sixes (§ 11);

moreover, each transformation of a *triad of associated σ-transformations*, i.e. each of the three σ-transformations which correspond to three associated double-sixes, is the transform of one of the other two by means of the third: so that the sub-group of $\mathfrak{G}$ defined by any two of them contains the third, and is simply isomorphic with the symmetric group of degree 3. Hence:

Two σ-transformations have a product of order 2 *or* 3, *according as they are or are not permutable; the sub-group of $\mathfrak{G}$ defined by them is respectively of order* 4 *and simply isomorphic with the trirectangular group* (*Vierergroup*), *or of order* 6 *and simply isomorphic with the symmetric group of degree* 3, *the complete sets of these two types including* $\frac{1}{2}.36.15 = 270$ *and* $\frac{1}{6}.36.20 = 120$ *groups respectively.*

We obtain at once the following relations (e) among the 36 σ-transformations, those of the first line expressing the involutory character of these transformations, and those of the successive lines giving in order the symbolical interpretation of the facts graphically represented in § 11 by Figs. 32–7.

$$\begin{aligned}
&[\alpha\beta]^2 = [\alpha\beta\gamma]^2 = 1,\\
&[\alpha\beta].[\alpha_1\gamma] = [\alpha_1\gamma].[\alpha\beta],\\
&[\alpha\beta\gamma].[3\gamma] = [3\gamma].[\alpha\beta\gamma], \text{ etc.},\\
&[\alpha\beta\gamma].[\alpha_1\beta_1\gamma] = [\alpha_1\beta_1\gamma].[\alpha\beta\gamma], \text{ etc.},\\
&[\alpha\beta_2].[\alpha\beta_3] = [\alpha\beta_3].[\alpha\beta_1] = [\alpha\beta_1].[\alpha\beta_2],\\
&[\alpha\beta\gamma_1].[\alpha\beta\gamma_2] = [\alpha\beta\gamma_2].[3\gamma_3] = [3\gamma_3].[\alpha\beta\gamma_1], \text{ etc.},\\
&[\alpha_2\beta_2\gamma_2].[\alpha_3\beta_3\gamma_3] = [\alpha_3\beta_3\gamma_3].[\alpha_1\beta_1\gamma_1] = [\alpha_1\beta_1\gamma_1].[\alpha_2\beta_2\gamma_2].
\end{aligned}\tag{e}$$

14. Two permutable double-sixes, δ and δ' say, have in common two pairs of conjugate lines of their sextuplets (§ 11); from the graphical representation we see, moreover, that the 2 σ-transformations attached to them, say ω and ω' respectively, are such that ω', for instance, transforms into itself each of the 2 sextuplets of δ, interchanging in it the 2 lines which also belong to δ' and leaving unchanged each of the 4 others. The 15 σ-transformations permutable with ω (§ 13) perform all the 15 possible transpositions among the 6 pairs of conjugate lines of the 2 complementary sextuplets constituting δ, and clearly define a group, of order $6! = 720$, simply isomorphic with the symmetric group of degree 6; by means of a convenient product of such σ-transformations, it is possible to perform any given substitution among these 6 pairs of lines.

A transformation of $\mathfrak{G}$ leaving unaltered the 12 lines of a double-six is necessarily the identical transformation, since it must also leave

unaltered the remaining 15 lines, each of which is the only one which is incident with a given 4 of the former 12. It follows that an operation of $\mathfrak{G}$ transforming into itself one of (and therefore both) the sextuplets of δ is expressible as a product of σ-transformations permutable with ω; and any operation of $\mathfrak{G}$ interchanging those two sextuplets can be obtained by multiplying such a product by ω. Hence:

The group $\mathfrak{G}$ contains a complete set of 36 *conjugate sub-groups of index* 36 (*i.e. of order* 1,440)*; each of them consists of all the operations of $\mathfrak{G}$ transforming one of the* 36 *double-sixes into itself, and is simply isomorphic with the direct product of two symmetric groups of degrees* 2 *and* 6.†

The only transformations of $\mathfrak{G}$ which leave unchanged each of the 15 lines residual to δ are identity and ω; hence the operations of $\mathfrak{G}$ which transform such a set of 15 lines into itself make up the group of order 1,440 (of the above type) defined by δ, but they induce only 720 distinct substitutions among the 15 lines; these determine, for their part, a group of substitutions among the 6 systems of 5 tritangent planes associated with the set of 15 lines (§ 10): and the last group is the *symmetric group of degree* 6, since a substitution among the 15 lines leaving unchanged each of these 6 systems also leaves unaltered each of the 15 tritangent planes (common to two of them), and, therefore, also each of the 15 lines (common to three tritangent planes). *The transformations of $\mathfrak{G}$ leaving unchanged one of these* 6 *systems of* 5 *tritangent planes induce the total group of* 120 *substitutions among its* 5 *tritangent planes,* since the transformations of $\mathfrak{G}$ which leave unchanged each of these 5 planes also leave unchanged each of the 6 systems, and consequently are only identity and ω.

15. If two permutable σ-transformations are given, then the remaining 34 σ-transformations can be distributed in three sets of 12, 16, and 6 operations, which are, respectively, permutable with none, one, or both of them. The 6 σ-transformations of a set of the last type are related among themselves in the same way as the 6 transpositions upon 4 elements (for instance, each of them is permutable with one and only one of the 5 others), so that they define a group of order $4! = 24$, simply isomorphic with the symmetric group of degree 4. *The group $\mathfrak{G}$ contains a complete set of* $\frac{1}{2}.36.15 = 270$ *conjugate sub-groups of this sort.*

The maximum number of mutually permutable σ-transformations is therefore 4; and we obtain in $\mathfrak{G}$ a *complete set of* $\frac{1}{6}.270.3 = 135$ *conjugate*

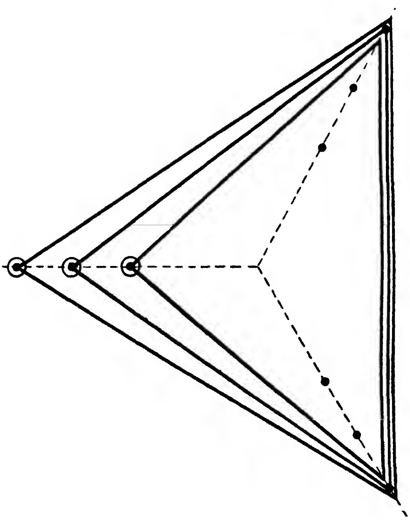
† In accordance with this, we have that $2.6!\,36 = 51{,}840$.

abelian sub-groups of order 16, each of which is determined by 4 mutually permutable σ-transformations, and consequently consists of the identity and 15 involutory transformations. The lines involved in 4 mutually permutable double-sixes are only 24 in number, being all the 27 lines except the 3 of a tritangent plane, and each of them belongs to exactly 2 of these 4 double-sixes. There are 8 σ-transformations non-permutable with (3 and therefore also with the fourth of) 4 given mutually permutable σ-transformations; the former are transformed transitively into themselves by the abelian group defined by the latter, one of them (and consequently all the 8) remaining unchanged by exactly 2 transformations of that group: the identity and the product of its 4 σ-transformations.

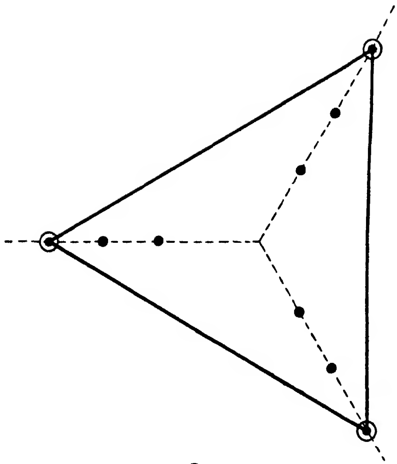
The number of the double-sixes which can be formed with 12 of the 24 lines residual to those of a given tritangent plane is 12. The 12 corresponding σ-transformations can be distributed (in a single manner) into 3 sets of 4 σ-transformations, such that 2 of the former 12 σ-transformations are or are not permutable, according as they belong to the same set or to different sets;† in the second case, the transform of one of these 2 σ-transformations by means of the other is a σ-transformation which belongs to the remaining set. Such a set of 12 transformations determine a *group of order* 51,840 : (45.6) = 192, given by all the transformations of $\mathfrak{G}$ leaving unaltered the 3 lines of a tritangent plane; in $\mathfrak{G}$ there is a *complete set of* 45 *conjugate sub-groups of this type*, each of which contains 3 of the abelian sub-groups of order 16 considered before.

All the above results can be established without difficulty by using our graphical representation. Thus, for instance, the 6 σ-transformations simultaneously permutable with the pair given by Fig. 32 are represented by Fig. 38; and we obtain for the 135 sets of 4 mutually permutable σ-transformations the representations of the different types given in Figs. 39–41, which account respectively for 27, 54, and 27 pairs of them. The 3 sets of 4 mutually permutable σ-transformations, which are connected in the above manner with the tritangent plane represented by Fig. 5, all have a representation of the type given by Fig. 40; and the properties enunciated for them, which are in an obvious relation with those of the 3 desmic tetrahedra associated with the tritangent plane (§ 9), are graphically evident.

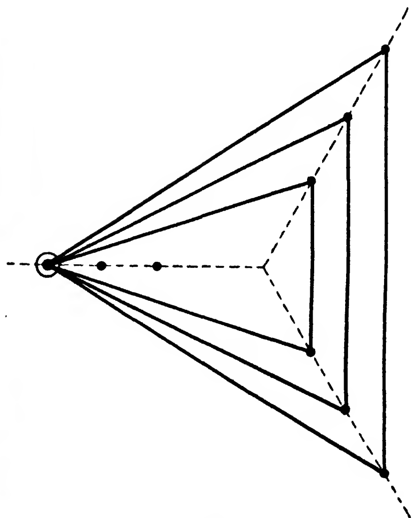
† The 4 σ-transformations of each of the 3 sets have as product the same transformation, which is the involutory transformation leaving fixed the 3 lines of the tritangent plane initially considered, and interchanging 2 further lines if (and only if) they are in a plane through one of these 3 lines.



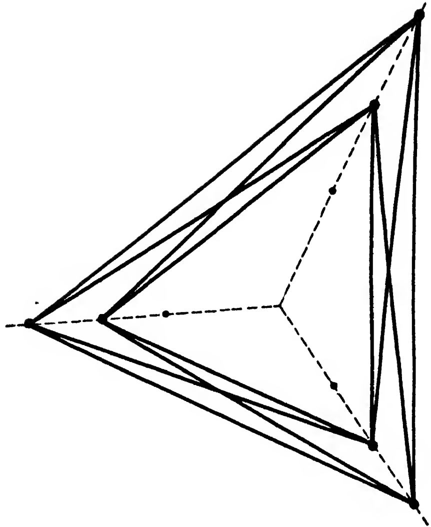
38



39



40



41

FIGS. 38–41

16. If 3 associated σ-transformations are given, then the remaining 33 σ-transformations can be distributed into two sets of 27 and 6 operations, which are respectively permutable with one or all of them. A set of the second type consists of 2 permutable triads of associated σ-transformations; so that from the 120 triads we can form 40 *sets of 3 permutable triads*. The 9 σ-transformations of such a set define one of the 40 groups simply isomorphic with the direct product of 3 symmetric groups of degree 3, which we have already considered in § 12 from another point of view.

Each of the 27 σ-transformations residual to the 9 elements of a set of 3 permutable triads is permutable with one and only one σ-transformation of each of the 3 triads, and is uniquely defined by the 3 elements of these triads with which it is permutable. The whole system of 27 σ-transformations is transformed into itself by each of the residual 9, and can be derived from one of its operations by properly transforming it by means of those 9 elements, as is at once suggested by our graphical representation.

V. Definition of $\mathfrak{G}$ as an abstract group

17. We shall now prove that:

The 36 σ-transformations constitute a complete set of conjugate operations and a system of generators of the group $\mathfrak{G}$; moreover, the equations (e) *(written at the end of* § 13) *constitute a complete system of relations connecting them.*

We begin by observing that the group defined by the 36 σ-transformations operates transitively upon the 36 double-sixes of F. Any two of these, δ and δ' say, are, in fact, either permutable or non-permutable (§ 11). In the first case, we consider one of the 12 double-sixes which are permutable with neither of them (§ 15), and the 2 double-sixes which are associated with it and, respectively, δ or δ': the 2 σ-transformations defined by these double-sixes are permutable, and their product interchanges δ and δ'. If, on the contrary, δ and δ' are non-permutable, then they are associate to a third double-six; this defines a σ-transformation, which again interchanges δ and δ'.

If an arbitrary transformation T of $\mathfrak{G}$ is given, we consider one, δ say, of the 36 double-sixes and its transform δ' by means of T. We have already seen how T can be expressed by a product of σ-transformations when δ' coincides with δ (§ 14); if, on the contrary, δ and δ' are distinct, we take a transformation S—given by a σ-transformation or by the product of 2 permutable σ-transformations—which inter-

changes δ and δ': then TS leaves δ unchanged, and is therefore expressible as a product of σ-transformations, so that the same is also true of T.

Since, in virtue of (e), the transform of any σ-transformation by means of a σ-transformation is again a σ-transformation, the first part of the theorem is thus established. In order to demonstrate the second part, we have to prove that each relation among the σ-transformations is a consequence of (e). Let us, in fact, make the contrary hypothesis, and suppose there is a relation among the σ-transformations which cannot be *shortened*, i.e. which cannot be reduced to one involving a smaller number of σ-transformations, by taking into account the equations (e). Such a relation can be written in the form

$$\omega\omega_1\omega_2...\omega_t = 1, \tag{1}$$

where ω, ω_1,..., ω_t are *distinct* σ-transformations, since, by virtue of (e), the σ-transformations are involutory and such that any two of them, ω' and ω'' say, satisfy a relation of the type

$$\omega'\omega'' = \omega''\omega''',$$

where ω''' is a suitably chosen σ-transformation. It follows that it is possible to substitute for (1) a relation still involving $t+1$ σ-transformations, the first s of which are still ω, ω_1,..., ω_{s-1}, while the $(s+1)$th is any other ω_r (with $1 \leqslant s \leqslant r \leqslant t$). We can therefore suppose (1) reduced to the form

$$\rho_1...\rho_h\,\omega\tau_1...\tau_k = 1 \qquad (h \geqslant 0,\ h+k = t), \tag{2}$$

where ρ_1,..., ρ_h are σ-transformations permutable with ω, τ_1,..., τ_k are σ-transformations non-permutable with ω, and it is not possible to increase the number of the former and decrease the number of the latter by applying (e).

The 2 sextuplets of the double-six δ defined by ω are transformed each into itself by ρ_1,..., ρ_h, interchanged by ω, and transformed into other sextuplets by τ_1,..., τ_k; in order that (2) may hold, we must therefore have $k \geqslant 2$. Should two of the τ_1,..., τ_k, for instance τ_1 and τ_2, be non-permutable, the σ-transformation ρ associated with them would be permutable with ω (§ 16); by virtue of (e) we should have $\tau_1\tau_2 = \rho\tau_1$, and we could increase in (2) the number of the transformations permutable with ω. This implies that the τ_1,..., τ_k can be supposed mutually permutable, and therefore k cannot be greater than 4 (§ 15); since for the validity of (2) their product has to transform δ into itself, we must have exactly $k = 4$, and the relations (e) yield

$$\omega^2 = \tau_i^2 = 1, \qquad \tau_i\tau_j = \tau_j\tau_i, \qquad \tau_i\omega\tau_i = \omega\tau_i\omega \quad (i, j = 1, 2, 3, 4) \tag{3}$$

$$\tau_1\tau_2\tau_3\tau_4\,\omega\tau_4\tau_3\tau_2\tau_1 = \omega, \tag{4}$$

as the left-hand side of (4) is a σ-transformation which can only be ω (§ 15).

From (3), (4) we deduce

$$\omega\tau_2\omega = \tau_2\omega\tau_2 = \tau_2\tau_1(\tau_1\omega\tau_1)\tau_1\tau_2 = \tau_2\tau_1\omega\tau_1\omega\tau_1\tau_2,$$

$$\begin{aligned}\omega\tau_3\omega = \tau_3\omega\tau_3 &= \tau_3(\tau_1\tau_2\tau_3\tau_4\omega\tau_4\tau_3\tau_2\tau_1)\tau_3 \\ &= \tau_2\tau_1(\tau_4\omega\tau_4)\tau_1\tau_2 = \tau_2\tau_1\omega\tau_4\omega\tau_1\tau_2,\end{aligned}$$

and therefore

$$\begin{aligned}\omega\tau_1\tau_2\tau_3\tau_4 &= \omega\tau_1\tau_2(\tau_2\tau_1\omega\tau_1\omega\tau_1\tau_2)(\tau_2\tau_1\omega\tau_4\omega\tau_1\tau_2)(\omega\tau_3\omega)(\omega\tau_2\omega)\tau_3\tau_4 \\ &= \tau_1\tau_4\omega\tau_1\tau_2\omega\tau_3\tau_2\omega\tau_3\tau_4 \\ &= (\tau_1\tau_4\omega\tau_4\tau_1)(\tau_2\tau_4\omega\tau_4\tau_2)(\tau_3\tau_4\omega\tau_4\tau_3),\end{aligned}$$

so that $\omega\tau_1\tau_2\tau_3\tau_4$ is expressed as the product of only three σ-transformations, and the relation (2) can consequently be shortened. This conclusion completes the proof of our theorem.

18. The relations (*e*) of § 13, connecting the 36 elements $[\alpha\beta]$ and $[\alpha\beta\gamma]$, define an *abstract group* $\mathfrak{H}$ (of order 51,840) which is simply isomorphic with $\mathfrak{G}$ (§ 17). The same group can be obtained more simply, as—according to our representation (§ 16)—each of those elements can be expressed as a function of 10 of them, 9 of which constitute a set of 3 permutable triads of associated elements; since only 6 among these 9 are independent, we can define $\mathfrak{H}$ as follows by means of 7 generators. Putting, for brevity,

$$\begin{matrix}[12] = p_1, & [22] = q_1, & [32] = r_1, \\ [11] = p_2, & [21] = q_2, & [31] = r_2,\end{matrix} \qquad [333] = s,$$

we have

$$\begin{matrix}[13] = p_1p_2p_1, & [23] = q_1q_2q_1, & [33] = r_1r_2r_1, \\ [i33] = p_isp_i, & [3i3] = q_isq_i, & [33i] = r_isr_i, \\ [3ij] = q_ir_jsr_jq_i, & [j3i] = r_ip_jsp_jr_i, & [ij3] = p_iq_jsq_jp_i,\end{matrix}$$
$$[ijl] = p_iq_jr_lsr_lq_jp_i \qquad (i, j, l = 1, 2),$$

and the relations (*e*) reduce to a form which we leave to the reader to write down explicitly. We confine ourselves to the remark that *the number of generators of* $\mathfrak{H}$ *can be reduced from* 7 *to* 6, since we have the relation

$$r_2 = sp_2q_2sp_1q_1r_1sr_1q_1p_1sq_2p_2s,$$

directly suggested by our graphical representation.

The study of $\mathfrak{H}$, and therefore of $\mathfrak{G}$, could easily be pursued along these lines. Thus, for instance, since each member of the relations (*e*) is quadratic in the σ-transformations, we have in $\mathfrak{G}$ an *invariant sub-*

group of index 2, given by all the elements of $\mathfrak{G}$ which are expressible as the product of an *even* number of σ-transformations; this sub-group, of order 25,920 (which is simply isomorphic with the group of the problem of the trisection of hyperelliptic functions of genus 2†), can therefore be defined by means of 5 generators connected by proper relations.

19. The group $\mathfrak{G}$ contains some proper sub-groups which, like $\mathfrak{G}$ itself, act transitively upon the 27 lines of F; one of them is, for instance, the group Γ considered in § 12. We shall, however, prove the following theorem, which, of course, reflects a property of the abstract group $\mathfrak{H}$ (defined in § 18), in connexion with its 27 conjugate sub-groups of index 27 corresponding to the groups of operations of $\mathfrak{G}$ which leave fixed a single line of F:

If a sub-group Γ of $\mathfrak{G}$ has any number of σ-transformations as generators and is transitive, then it must necessarily coincide with $\mathfrak{G}$.‡

We start from the remark that, if the σ-transformations belonging to Γ were permutable 2 by 2, their number would be $\leqslant 4$ and the corresponding double-sixes would involve at most 24 distinct lines (§ 15); the remaining lines would then be self-corresponding for each of those σ-transformations, and therefore also for each transformation of Γ, which is in contradiction with the supposed transitivity of Γ. Two at least of the σ-transformations of Γ must consequently be non-permutable, so that Γ contains the set of 3 associated σ-transformations determined by them; and we can suppose, without restriction, this set to be

$$[11], \qquad [12], \qquad [13]. \tag{5}$$

The group Γ has, moreover, in common with each of the sets

$$[21], \qquad [22], \qquad [23] \tag{6}$$

and

$$[31], \qquad [32], \qquad [33] \tag{7}$$

a number of transformations which can only be 0, 1, or 3. We now proceed to examine separately the 6 different cases that *a priori* can arise in this connexion; we shall thus show that only one of them is actually possible, at the same time proving the theorem.

† For this and further literature on the matter, cf. H. Burkhardt, 'Untersuchungen aus dem Gebiete der hyperelliptischen Modulfunctionen', *Math. Ann.*, vol. 41 (1893), pp. 313–43.

‡ This result has already been deduced by Todd from the complete enumeration, given by Coxeter, of sub-groups generated by reflection, of finite groups generated by reflections: cf. J. A. Todd, 'On the Topology of Certain Algebraic Threefold Loci', *Proceedings Edinburgh Math. Soc.* (II), vol. 4 (1935), pp. 175–84, § 4.

We remark as a preliminary that, if $[\alpha\beta\gamma]$ belongs to Γ, all the three transformations $[1\beta\gamma]$, $[2\beta\gamma]$, $[3\beta\gamma]$, which are transforms of it by means of the three transformations (5), belong to Γ: so that we can simply say that $[*\beta\gamma]$ belongs to Γ.

(i) If Γ does not contain any transformation (6) or (7), it cannot contain two distinct transformations $[\alpha\beta\gamma]$, $[\alpha\beta'\gamma']$ having either $\beta = \beta'$ or $\gamma = \gamma'$, since otherwise $[\alpha\beta\gamma].[\alpha\beta'\gamma'].[\alpha\beta\gamma]$—which is one of the transformations (7) or (6)—would belong to Γ. It follows that Γ can, at the most, contain $[*\beta_1\gamma_1]$, $[*\beta_2\gamma_2]$, $[*\beta_3\gamma_3]$ as further σ-transformations (where $\beta_1\beta_2\beta_3$ and $\gamma_1\gamma_2\gamma_3$ are two determined permutations of the numbers 1, 2, 3); but then the line $0\beta_1\gamma_1$ is self-corresponding for all the σ-transformations of Γ, and therefore is fixed for Γ, which is in contradiction with the transitivity of this group.

(ii) If $[21]$ is the only transformation that Γ has in common with (6) and (7), the only further σ-transformations belonging to Γ are of the type $[*1\alpha]$, $[*2\beta]$, $[*3\gamma]$. Of two sets of transformations like $[*2\beta]$ and $[*3\beta]$, either both or neither belong to Γ, since each of them is the transform of the other by means of $[21]$; but, with our assumption, $[*1\alpha]$ and $[*2\alpha]$, or $[*2\alpha_1]$ and $[*2\alpha_2]$ (with $\alpha_1 \neq \alpha_2$) cannot both belong to Γ. Besides (5) and $[21]$, Γ can, therefore, at the most contain the σ-transformations $[*1\alpha]$, $[*2\beta]$, $[*3\beta]$, where α, β assume two distinct values among the numbers 1, 2, 3; but then the line 01α is fixed for Γ, and we conclude as before.

(iii) If $[21]$ and $[31]$ are the only transformations that Γ has in common with (6) and (7), Γ must contain some of the transformations $[*1i]$, $[*i1]$ $(i = 2, 3)$, in order that the line 011 is not united. Let us suppose, for instance, that Γ contains $[*12]$, and therefore also $[*13]$, which is the transform of $[*12]$ by means of $[31]$. It is immediately seen that, with our assumptions, Γ does not contain any of the transformations $[*ij]$ (with $i, j = 2, 3$) or $[*11]$; it follows that neither $[*21]$ nor $[*31]$ can belong to Γ, since, for instance, the transform of $[121]$ by $[212]$ is $[333]$, which is not in Γ. The only σ-transformations of Γ are, then, (5), $[21]$, $[31]$, $[*12]$, $[*13]$; these transform into itself the pair of lines 021, 031, which is in contradiction with the transitivity of Γ.

(iv) If Γ contains all the transformations (6) but no transformation (7), it must contain at least one σ-transformation of the type $[\alpha\beta\gamma]$, since otherwise Γ would change into itself each of the Steiner sets represented by the 3 determinants of the table (I), and could therefore not be transitive. If, for instance, Γ contains $[333]$, it must also contain all the transformations $[\alpha\beta3]$, which are its transforms by (5) and (6);

no other σ-transformation can belong to Γ, since, for instance, the transform of $[\alpha\beta 2]$ by $[\alpha\beta 3]$ is $[31]$, which does not belong to Γ. All the σ-transformations of Γ are consequently permutable with $[33]$, so that they—and therefore also Γ—must transform into itself the set of 12 lines of the double-six $\{33\}$ inherent to $[33]$; which is in contradiction with the transitivity of Γ.

(v) If we suppose that Γ contains all the transformations (6) and has in common with (7) only one transformation, say $[33]$, we see, as in (iv), that Γ must contain at least one of the $[\alpha\beta\gamma]$, and consequently all those with the same γ. We must, moreover, have $\gamma = 3$; since, if, for instance, $[\alpha\beta 1]$ belongs to Γ, so also does $[\alpha\beta 2]$ (transform of $[\alpha\beta 1]$ by $[33]$) and $[\alpha\beta 3]$ (since, e.g., $[333]$ is the transform of $[111]$ by $[222]$), so that Γ would also contain all the transformations (7). Now again all the σ-transformations of Γ are permutable with $[33]$, and we conclude as before.

(vi) If, finally, Γ contains all the transformations (5), (6), (7), we see, as in (iv), that at least one of the 27 transformations $[\alpha\beta\gamma]$ must belong to Γ. But then all these 27 transformations belong to Γ, since they are the transform of one of them by convenient products of transformations (5), (6), (7); and we conclude (§ 17) that Γ coincides with $\mathfrak{G}$.

VI. The continuous variation of the non-singular cubic surfaces and the double-sixes

20. We shall now prove that,

If we consider (in [3] or upon any non-singular V_3) a rational ∞^1 system Ω of cubic surfaces, the generic F of which is non-singular while all the singular surfaces of Ω are uninodal, then the group Γ of substitutions upon the 27 lines of F induced by the circulations of F in Ω coincides with $\mathfrak{G}$ if, and only if, the ruled surface R generated by the 27 lines of F is irreducible.†

Each circulation of F in Ω is equivalent to a product of loops surrounding one of the singular surfaces of Ω, by virtue of the rationality of this system. The transformation of Γ which corresponds to such a loop is a σ-transformation. Indeed, when F describes a given path γ and tends to a nodal surface F_0, each line of F_0 through its double point is the limit of two distinct lines of F; the 6 pairs of lines which we thus have on F are the pairs of conjugate lines of a double-six:‡ and

† The case in which Ω is a pencil of prime sections of a general cubic primal of [4] has already been considered by Todd (loc. cit., § 4).

‡ This well-known result is here completely proved in § 23.

the σ-transformation inherent to this is the transformation of Γ which corresponds to the loop round F_0 determined by γ. Since the irreducibility of R is equivalent to the transitivity of Γ, the above-enunciated result is an immediate consequence of § 19.

From this result follows at once the possibility (stated already in § 3) of *generating the whole group* $\mathfrak{G}$ *by means of circulations of* F *in* [3] or, if we prefer, by means of circulations of F in a generic pencil.

21. The last corollary can also be established in a very elementary way, using the exact specification of the so-called *double-six theorem* which we now explain.

Let us consider in [3] a line b_6 incident to 5 lines a_i $(i = 1, 2,..., 5)$, and determining with them the points A_i and the planes α_i. In order that the 6 lines $(a_1,..., a_5, b_6)$ belong to a double-six, the following conditions are evidently necessary:

(i) Both the 5 points A_i and the 5 planes α_i are distinct, and such that, if i_1, i_2, i_3, i_4 are any 4 distinct values among the numbers 1, 2,..., 5,

$$(A_{i_1} A_{i_2} A_{i_3} A_{i_4}) \neq (\alpha_{i_1} \alpha_{i_2} \alpha_{i_3} \alpha_{i_4}).\dagger$$

(ii) The 5 lines a_i have b_6 as their only common transversal, and each of them is distinct from b_6.

Conversely, *if these conditions* (i), (ii) *hold, then* $(a_1...a_5, b_6)$ *belong to one (and only one) double-six, lying on a well-defined non-singular cubic surface.*

These 6 lines impose, in fact, at the most 19 independent conditions on the cubic surfaces, so that we have always at least one cubic surface through them. Such a surface is necessarily irreducible and non-ruled, by virtue of (i) and (ii), and must contain each of the further lines $b_1 b_2 ... b_5$ intersecting 4 of the a_i's; $b_1 b_2 ... b_5$ are 5 distinct and well-defined lines, which also are distinct from b_6, as well as from the a_i's, owing to (i) and (ii). Consequently there cannot be two distinct cubic surfaces through $(a_1 ... a_5, b_6)$, since otherwise they would meet in a curve of the 11th order without having a part in common. We conclude that there is one and only one cubic surface through $(a_1 ... a_5, b_6)$, F say, and that it is non-singular; indeed, 5 skew lines belonging to a singular irreducible and non-ruled cubic surface are always such that at least one tetrad of them has a unique common transversal, as is apparent by projecting the surface stereographically from its double point. From the incidence properties of the 27 lines of F we deduce

† This relation is a consequence of the fact that $a_{i_1} a_{i_2} a_{i_3} a_{i_4}$ must have two (and only two) distinct common transversals.

at once the existence of a line a_6 intersecting $b_1 \dots b_5$, and therefore constituting a double-six with the 11 lines considered above.

The 38-dimensional real manifold Θ, which represents the ordered sets of 6 complex lines $(a_1 \dots a_5, b_6)$ satisfying the previous conditions, is manifestly connected, and related to the manifold Σ considered in § 3 in a (51,840, 1)-correspondence: whence the last proposition of § 20 follows immediately. We see, moreover, that

The Poincaré group $\mathfrak{P}_\Sigma$ *of* Σ *contains a self-conjugate sub-group simply isomorphic with the Poincaré group* $\mathfrak{P}_\Theta$ *of* Θ, *the quotient group* $\mathfrak{P}_\Sigma/\mathfrak{P}_\Theta$ *being simply isomorphic with* $\mathfrak{G}$.†

Similar considerations can be applied to the Poincaré groups of the 5 19-dimensional manifolds Σ_i, introduced later on in connexion with Σ (§ 24), and the sub-groups Γ_i of $\mathfrak{G}$ determined in section IX.

When a non-singular cubic surface F degenerates into 3 planes π_1, π_2, π_3 (as in section I), we have on F a *fundamental triad of complementary Steiner sets*, given by the 3 sets of lines of F [represented by Table (I) of § 4] whose limits belong to a single plane π_i; such a triad characterizes the corresponding graphical representation for the 27 lines of F, but for an inessential transformation of the group Γ considered in § 12. Since $\mathfrak{G}$ operates transitively among the 40 triads of complementary Steiner sets, we deduce, taking also into account the final result of § 20, that:

In the complex field a non-singular cubic surface can be made to degenerate into 3 *independent planes in* 40 *essentially distinct ways, leading to* 40 *essentially distinct graphical representations for its* 27 *lines.*

22. Let us consider a double-six δ on a non-singular cubic surface F; then the 27 lines of F can be represented in Schläfli's notation by putting

$$\delta = \begin{pmatrix} a_1\, a_2\, a_3\, a_4\, a_5\, a_6 \\ b_1\, b_2\, b_3\, b_4\, b_5\, b_6 \end{pmatrix}$$

[where a_i, b_i $(i = 1, 2, \dots, 6)$ are the 6 pairs of conjugate lines of the complementary sextuplets constituting δ (§ 7) taken in any order], and by denoting by c_{ij} $(= c_{ji})$ the line of F incident with a_i, a_j, b_i, b_j $(i, j = 1, 2, \dots, 6,\ i \neq j)$. We shall, moreover, represent by L_{ij} $(\neq L_{ji})$ and by λ_{ij} $(\neq \lambda_{ji})$ the point and the plane determined by a_i, b_j. From our graphical representation it follows at once that c_{ij}, c_{rs} are incident if and only if both the indices i, j are distinct from r, s. We say that

$$(L_{13}\, L_{14}\, L_{15}\, L_{16}) = (L_{32}\, L_{42}\, L_{52}\, L_{62}). \tag{1}$$

† Concerning the notion of the Poincaré group, see, for instance, S. Lefschetz, 'Topology' (*Amer. Math. Soc.*, 1930), p. 82.

In fact the line $L_{1r}L_{s2}$ ($r, s = 3, 4, 5, 6, r \neq s$), join of a point of a_1 with a point of b_2, intersects c_{12}, which is in the plane λ_{12} of a_1 and b_2; and, for a similar reason, it intersects $c_{sr} = c_{rs}$, so that it must go through the point in which this line intersects the plane λ_{12}, that is, through the point common to c_{12} and c_{rs}. Through the same point goes likewise $L_{1s}L_{r2}$; hence the two quadruplets of points of a_1 and b_2 appearing in the 1st and 2nd member of (1) are projective, since they have the same projection on to c_{12} from L_{32} and L_{13} respectively.

By replacing in (1) the second of those quadruplets by its projection from the axis b_1 (skew to b_2), we obtain

$$(L_{13}\,L_{14}\,L_{15}\,L_{16}) = (\lambda_{31}\,\lambda_{41}\,\lambda_{51}\,\lambda_{61}); \tag{2}$$

and this relation, together with all those deducible from it by applying any substitution upon the indices 1, 2,..., 6,† is an immediate consequences of the following theorem of Schur.‡

The 6 pairs of conjugate lines of a double-six are mutually polar with respect to a quadric. This quadric intersects each line of the double-six in its parabolic points, and contains the 24 lines joining all the pairs of parabolic points which belong to two conjugate lines.

In order to prove this theorem, we consider the points M_{ij}, M_{ji}, N_{ij}, N_{ji} intersected by c_{ij} ($= c_{ji}$) on a_i, a_j, b_j, b_i respectively ($i, j = 1, 2,..., 6, i \neq j$), and we fix a pair of conjugate lines of δ, a_6 and b_6 say. The cubic surface F determines on a_6 an involution of pairs of points having the same tangent plane; the 5 pairs of points L_{6h}, M_{6h} ($h = 1, 2,..., 5$) correspond in it, since F is touched in such a pair by the plane λ_{6h}, and the double points of that involution are the parabolic points of a_6 (§ 6), R_1 and R_2 say. Likewise the parabolic points of b_6, S_1 and S_2 say, are the double points of an involution in which L_{h6} and N_{h6} correspond, since F is touched in such a pair by the plane λ_{h6}.

If Q' is any irreducible quadric through

$$R_1S_1, \qquad R_1S_2, \qquad R_2S_1, \qquad R_2S_2, \tag{3}$$

then the lines a'_i, b'_j, c'_{ij} ($= c'_{ji}$) polar with respect to Q' of a_i, b_j, c_{ij} ($i, j = 1, 2,..., 6$) constitute a double-six δ', for which we adopt the same notation, with dashes, as for δ. Obviously, then,

$$a'_6 = b_6, \qquad b'_6 = a_6,$$

† Such relations are established analytically in § 8 of Henderson's monograph quoted in the Preface.

‡ Several bibliographical indications on this theorem can be found in the note (20) on p. 1452 of Meyer's article quoted in the Preface.

and, moreover,

$$L'_{6h} = L_{h6}, \quad L'_{h6} = L_{6h}, \qquad \lambda'_{6h} = \lambda_{h6}, \quad \lambda'_{h6} = \lambda_{6h} \quad (h = 1, 2,..., 5);$$

in fact, for instance, the point L_{6h} is on a_6, so that its polar plane with respect to Q' is the plane joining b_6 to the point M_{6h} (harmonic conjugate of L_{6h} with respect to R_1, R_2), that is, it coincides with λ_{h6}: and, since L_{6h} is the point common to a_6, b_h, it follows that λ_{h6} is the plane joining a'_6, b'_h, which we have called λ'_{6h}. We see consequently that

$$c'_{h6} = \lambda'_{6h}\lambda'_{h6} = \lambda_{h6}\lambda_{6h} = c_{h6}, \qquad M'_{6h} = a'_6 c'_{h6} = b_6 c_{h6} = N_{h6},$$
$$N'_{h6} = b'_6 c'_{h6} = a_6 c_{h6} = M_{6h}.$$

Hence the cubic surface F' containing δ' has the lines c_{16}, c_{26}, c_{36}, c_{46}, c_{56} in common with F, and it touches this surface along the lines a_6, b_6, as both F and F' are touched by λ_{6h} at L_{6h}, M_{6h}, and by λ_{h6} at L_{h6}, N_{h6} $(h = 1, 2,..., 5)$. It follows that, when the quadric Q' describes the pencil determined by the base-lines (3), the cubic surface F' varies in a pencil containing F. It is clearly possible to choose Q' in such a position Q that the corresponding surface F' coincides with F: then δ' coincides with δ, and Q (which is uniquely defined by δ) is clearly the Schur quadric of this double-six.

We remark that *a homography interchanging conjugate lines of* δ *cannot exist*; in fact, for instance, we have

$$(L_{31}\, L_{41}\, L_{51}\, L_{61}) \neq (\lambda_{31}\lambda_{41}\lambda_{51}\lambda_{61})$$

by virtue of § 21, (i), so that from (2) we deduce

$$(L_{13}\, L_{14}\, L_{15}\, L_{16}) \neq (L_{31}\, L_{41}\, L_{51}\, L_{61}). \tag{4}$$

We add that, since, by § 6, the parabolic points of a_i, b_j, c_{ij} are the 3 pairs of opposite vertices of a plane quadrilateral, and the former 2 pairs of points belong to Q, it follows that the latter pair consists of 2 points conjugate with respect to Q. Hence:

The 15 *pairs of parabolic points of the* 15 *lines of* F *residual to a double-six are conjugate with respect to the Schur quadric of the double-six.*

Since 2 complementary triplets of lines belong to 2 distinct (non-permutable) double-sixes (§ 11), we have, moreover, that:

The 6 *pairs of parabolic points of the lines of any two complementary triplets are on a twisted quartic of the* 1*st kind.*

III
THE REAL CUBIC SURFACES

VII. Existence, in the projective space, of five continuous systems of non-singular real cubic surfaces

23. The geometrical entities that we consider in this chapter are all supposed to be *real*, unless the contrary is stated. We begin by arriving, in a very simple way, at the projective classification of the non-singular cubic surfaces.

We start from the consideration of the general singular cubic surfaces, i.e. of the *cubic surfaces* Φ having a unique singular point, which is a (real) node or an isolated double point Q through which there are 6 distinct, real or complex, lines of Φ. Such a surface contains 15 other lines, which are its further intersections with the planes joining these 6 lines 2 by 2, and have the same incidence relations as the 15 lines of one of the 36 sets considered at the end of § 8.

Taking into account the fact that, if Φ contains a complex line, it must also contain the conjugate one, and that a complex line is said to be of the 1st or 2nd kind according as it is incident with, or skew to, its conjugate (i.e. according as it contains one real point or none), we see at once that Φ—in so far as its lines are concerned—*can only be of one of the 4 types specified by the following table.*

Singular cubic surface	*Number of its lines through Q*		*Number of its remaining lines*		
	Real	*Conjugate of the 1st kind*	*Real*	*Conjugate of the 1st kind*	*Conjugate of the 2nd kind*
Φ_1	6	..	15	..	..
Φ_2	4	2	7	..	8
Φ_3	2	4	3	4	8
Φ_4	..	6	3	12	..

We remark now that each non-singular cubic surface can obviously be continuously deformed into a general singular one remaining in the real field (for instance, moving in a real pencil); and, moreover,

When a non-singular cubic surface F tends to a general singular one Φ, 12 of its lines—constituting a double-six—tend to the 6 lines through the double point Q of Φ, in such a way that any two corresponding lines of the two sextuplets of this double-six tend to coincide in one line through Q.†

† This result also holds in the complex field, since the proof given here remains valid in it.

In fact, in virtue of § 2, the 15 lines of F having as limit the 15 lines of Φ not going through Q must have the same incidence relations as these lines, so that their residual 12 lines constitute a double-six (§ 8); any 2 corresponding lines of this double-six tend to coincide in a line through Q, since they are both incident to the same 5 among the 15 lines initially considered, and these incidence relations are preserved by the passage to the limit (§ 2).

The double-six of F having as limit the set of 6 lines of Φ through Q is necessarily self-conjugate, this being true of the limiting set, so that its 2 sextuplets are either both self-conjugate or mutually conjugate; a line of Φ through Q being the limit of two corresponding (and therefore skew) lines of those sextuplets, it follows that in the two cases respectively:

(i) a real line through Q is the limit either of 2 real or of 2 complex-conjugate lines of the 2nd kind;

(ii) a complex line through Q is the limit of 2 complex (non-conjugate) lines either of the 2nd or of the 1st kind.

We can therefore conclude that:

A non-singular cubic surface—in so far as its lines are concerned—can only be of one of the 5 types specified by the following table:

Non-singular cubic surface	*Number of its lines which are*		
	Real	*Conjugate of the 1st kind*	*Conjugate of the 2nd kind*
F_1	27	..	..
F_2	15	..	12
F_3	7	4	16
F_4	3	12	12
F_5	3	24	..

A singular surface Φ_i ($i = 1, 2, 3, 4$) can only be the limit of a non-singular one either of the type F_i or of the type F_{i+1}.

24. The well-known linear representation of the (real) cubic surfaces of [3] by the (real) points of a [19] (a representation which we have already considered in § 3 in the complex field) enables us to show the last theorem of § 23 in a more suggestive light, and to draw other results from it.

First of all we have in [19] an algebraic primal V, of order 32, locus of the image-points of the singular surfaces of [3]; and the simple points of this primal form 4 18-dimensional (open) manifolds V_i ($i = 1, 2, 3, 4$), without points in common, which are the loci of the image-points of

the surfaces Φ_i (cf. §§ 23 and 25). On V we also consider the four 17-dimensional (open) manifolds W_i ($i = 1, 2, 3, 4$), the points of which represent the cubic surfaces having a single double point, through which there are 6 distinct lines—$i-1$ pairs of which are complex-conjugate—situated in 2 (either real or complex-conjugate) distinct planes; we denote by Ψ_i a cubic surface of this type. In virtue of what will follow (in § 25), $W_1+W_2+W_3+W_4$ is an ordinary cuspidal locus of V, its points being the simple points of an irreducible algebraic 17-dimensional variety; it is, moreover, evident that W_i cuts the neighbourhood of each of its points upon V in 2 18-dimensional cells, both of which belong to the closure $\bar{V}_i$ of V_i.

In consequence of § 23, the open manifold $[19]-V$ (which is the section by [19] of the connected manifold Σ considered in § 3) consists of 5 regions Σ_j ($j = 1, 2, 3, 4, 5$), loci of the image-points of the surfaces F_j, no two of these regions having points in common; and *the complete contours of* $\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4, \Sigma_5$ *are, respectively,* $\bar{V}_1$, $\bar{V}_1+\bar{V}_2$, $\bar{V}_2+\bar{V}_3$, $\bar{V}_3+\bar{V}_4$, $\bar{V}_4$.

By using the stereographic projection of the surfaces Φ_i from their double points, we see that each of the manifolds V_l (for $l = 1, 2, 3$) is connected; since such a manifold has only simple points, it presents two well-defined sides in [19]: by virtue of § 23, one of these sides belongs to Σ_l and the other belongs to Σ_{l+1}. The manifold V_4 is, on the contrary, disconnected, and consists of 2 distinct 18-dimensional sheets, V_4' and V_4'', the points of which represent the surfaces Φ_4 having as singular point Q a node or an isolated double point; these two sheets are, however, connected by means of W_4 in such a way that $V_4'+W_4+V_4''$ is a connected manifold presenting in [19] two well-defined sides, one of which belongs to Σ_4 and the other to Σ_5.†

We shall now prove that *the region* Σ_5 *is connected.* In fact, any two, M, N, of its points can be joined to two points R, S of V_4 (namely, either of V_4' or of V_4'') by two paths MR, NS which, save for their ends R, S, lie in Σ_5;‡ these can be joined by a path RS lying entirely in $V_4'+W_4+V_4''$: and two points R', S' respectively near to R, S, and belonging to MR, NS, can be joined by a path $R'S'$ near to RS and wholly situated in Σ_5, so that the path $MR'+R'S'+S'N$ joins M to N within Σ_5.

† W_4 belongs in fact to both $\bar{V}_4'$ and $\bar{V}_4''$, and, moreover, the neigbourhoods of a point of the former upon the latter consist of two non-singular 18-dimensional cells, having in common a non-singular cell of their contours situated on W_4.

‡ For instance, a generic line r through M which intersects V contains only simple points of this primal; and, by virtue of § 23, the first of these points which we meet as we move on r starting from M in an arbitrary direction must belong to V_4.

The connected region Σ_4^* formed by the points of Σ_4 which can be connected within Σ_4 to points of V_4 must coincide with Σ_4 itself, since otherwise the only 18-dimensional contour of Σ_4^* would be $\bar{V}_4$, so that Σ_4^* would at the most differ from $[19]-\bar{\Sigma}_5$ by varieties of dimension < 18, which is not possible, as Σ_4^* is a component of Σ_4. *The region* Σ_4 *is therefore connected*; similarly we reach the same conclusion for Σ_3, Σ_2, and Σ_1 in succession.

The interpretation of these results in [3] is that:

The non-singular cubic surfaces of [3] *form precisely* 5 *distinct continuous systems; in other words, two surfaces of the same type (according to the classification of* § 23) *can always be continuously deformed into one another, in such a way that all the intermediate surfaces are of the same type as the two extreme positions.*

25. Let us consider in [3] a real or complex pencil of cubic surfaces, one, Φ, of which has a double point Q belonging to 6 distinct lines of Φ (so that Q cannot be uniplanar); we call $\mathfrak{C}$ the base curve of the pencil, and K the (possibly reducible) cone touching Φ at the point Q. It is then not difficult to show† that Φ, among the 32 (real or complex) singular surfaces of the pencil, counts a certain number of times, this number being:

(i) *exactly* 1 if K is irreducible and $\mathfrak{C}$ does not pass through Q;

(ii) *at least* 2 if K is irreducible and $\mathfrak{C}$ passes through Q;

(iii) *exactly* 2 if K is reducible and $\mathfrak{C}$ does not pass through Q;

(iv) *exactly* 3 if K is reducible and $\mathfrak{C}$ passes twice through Q, without touching at this point the double line of K;

(v) *at least* 4 if K is reducible and either $\mathfrak{C}$ passes twice through Q touching at this point the double line of K, or $\mathfrak{C}$ passes 3 times at least through Q.

We now suppose in particular that Φ is real, and consider its image J in [19] (§ 24). If K is irreducible, we have from (i), (ii) that J is a simple point of V, and that the prime touching V at J represents the ∞^{18} linear system consisting of the cubic surfaces of [3] through Q. If, on the contrary, K consists of two distinct planes intersecting along a line l, (iii), (iv), and (v) show that J is a simple point of a 17-dimensional cuspidal variety of V, that the prime touching V at J still represents the ∞^{18} linear system consisting of the cubic surfaces of [3] through Q, and that the secundum touching the cuspidal variety at J represents the ∞^{17} linear system consisting of the cubic surfaces of [3] touching l at Q.

† Cf. the Appendix III at the end of the volume.

In both cases the neighbourhood of the point J in [19] is cut by V along two cells, which belong to two different regions Σ_i. But, while in the first case every line through J which does not touch V at this point crosses both these cells, in the second case such a line crosses only one of the cells, which—as is easily proved†—belongs to Σ_i if J belongs to $\overline{V}_i$.

The last result of § 23 can therefore be put more exactly by saying that:

If an ∞^1 continuous system Θ of cubic surfaces contains a singular surface of the type Φ_i or of the type Ψ_i ($i = 1, 2, 3, 4$), but the envelope of the former does not contain the double point of the latter, then any sufficiently small neighbourhood in Θ of the surface considered either consists further of surfaces of both the types F_i and F_{i+1}, or only contains further surfaces of the type F_i respectively.

26. From §§ 24, 25 we easily deduce that, if a path joining in [19] a point of Σ_i with a point of Σ_j ($1 \leqslant i < j \leqslant 5$) has no point in common with V outside $V_1+V_2+V_3+V_4+W_1+W_2+W_3+W_4$, then the path must intersect each of the manifolds $V_i+W_i,\ldots, V_{j-1}+W_{j-1}$ at least once. One of the simplest consequences of this remark is the following theorem.

In the pencil determined by any 2 non-singular cubic surfaces, of the types F_i and F_j ($1 \leqslant i < j \leqslant 5$), there are at least $2(j-i)$ distinct real singular surfaces having a double point, origin of 6 distinct (real or complex) lines, with a possible exception only in the case in which the pencil contains some surface having some more complicated singularity.

VIII. The graphical representations and properties of the 5 projective types of real non-singular cubic surfaces

27. We say that a real line r of a (real) non-singular cubic surface F is *elliptic* or *hyperbolic*, in conformity with the type of the involution ρ determined on r by the conics which are the further intersections of F with the planes through r, that is, according as r has two conjugate complex or real parabolic points (§ 6). If a real point A of r moves upon this line in one direction, its tangent plane α turns around r; and α reverses the direction of its rotation if, and only if, A crosses a double point of ρ. It follows that:

While A describes r once, the corresponding tangent plane α describes twice either the whole pencil of axis r or a complete angle of this pencil

† For instance, by using the stereographic projection of Φ from the point Q.

(limited by the two parabolic planes through r), according as r is elliptic or hyperbolic.

Of particular interest are the 5 pairs of (real or complex) points of ρ, which correspond to the 5 positions of α given by the tritangent planes of F containing r. If one of these planes is real and intersects F further in 2 conjugate complex lines not in a pencil with r, i.e. if it touches F at a (real) elliptic point, then the corresponding points of r are conjugate complex, so that ρ is hyperbolic. Hence:

A sufficient (but not necessary) condition that a real line r of F be hyperbolic, is given by the existence of a plane through r touching F at an elliptic point.

This condition is equivalent to the existence of a plane through r which has no further real intersection with F; according as it is verified or not, we call the hyperbolic line r *of the 2nd* or *of the 1st kind.*

It is obvious that, during any continuous variation of F in the real field throughout which it remains non-singular, the property of one of its lines of being real and elliptic or hyperbolic of the 1st or 2nd kind, as well as of being complex of the 1st or 2nd kind, is always preserved. The numbers of lines of F of the above 5 different types, and also the geometrical configurations formed by all the lines of a single type, are *peculiar to each type of cubic non-singular surface.* Similar considerations can, of course, be applied to the classification in the real domain of the sets of lines of F of any kind; the several questions which thus arise in this connexion will later be given a very simple and suggestive answer, through the real specifications of our graphical representation for the 27 lines of a cubic surface, which we now proceed to study.

28. All the developments of sections I and II retain most of their validity if we remain in the real field; but—from the real point of view—there are many cases to be distinguished (stated in detail in § 29), since both the 3 planes π_1, π_2, π_3, constituting the (real) limiting surface F_0, and the 3 lines p_1, p_2, p_3, in which they intersect two by two, as well as the 3 points $P_{\alpha 1}$, $P_{\alpha 2}$, $P_{\alpha 3}$ existing upon each p_α of these lines which is real, can be either all real or one real and two conjugate complex.

In every case, a line $r_0 = P_{\alpha_1\beta}P_{\alpha_2\gamma}$ is the limit of a well-defined line r of F, of which (taking into account § 5) we know immediately the behaviour in relation to its conjugate. The line r is therefore

(i) *real*, if either $P_{\alpha_1\beta}$ and $P_{\alpha_2\gamma}$ (as well as p_{α_1} and p_{α_2}) are real, or $P_{\alpha_1\beta}$ and $P_{\alpha_2\gamma}$ (and therefore also p_{α_1} and p_{α_2}) are complex-conjugate;

(ii) *complex of the* 1*st kind* if either just one of the points $P_{\alpha_1\beta}$, $P_{\alpha_2\gamma}$ and just one of the lines p_{α_1}, p_{α_2} are real, or the 2 points are complex and the 2 lines are real, or the 2 points are complex non-conjugate and the 2 lines are conjugate complex;

(iii) *complex of the* 2*nd kind*, in the remaining cases, that is, if either just one of the points $P_{\alpha_1\beta}$, $P_{\alpha_2\gamma}$ and both the lines p_{α_1}, p_{α_2} are real, or both the points and just one of the lines are complex.

When F tends to F_0, the involution ρ (obtained on r as in § 27) has a well-defined and non-degenerate limit. The limiting involution is that determined upon r_0 by the 3 pairs of points intersected on this line by the 3 pairs of opposite sides of the quadrangle having as vertices the points $P_{\alpha_1\beta_1}$, $P_{\alpha_1\beta_2}$, $P_{\alpha_2\gamma_1}$, $P_{\alpha_2\gamma_2}$ (which are the points P which belong to p_{α_1}, p_{α_2} besides $P_{\alpha_1\beta}$, $P_{\alpha_2\gamma}$), so that $P_{\alpha_1\beta}$, $P_{\alpha_2\gamma}$ is always a pair of that involution; in fact the two pairs of opposite sides distinct from p_{α_1}, p_{α_2} are the limits, for $F \to F_0$, of the further intersections of F with two of its tritangent planes through r (§ 6). We therefore have that r, in the case in which it is real, is:

(i) *elliptic* if $P_{\alpha_1\beta}$, $P_{\alpha_2\gamma}$ are real, and just one of the pairs $P_{\alpha_1\beta_1}$, $P_{\alpha_1\beta_2}$ and $P_{\alpha_2\gamma_1}$, $P_{\alpha_2\gamma_2}$ consists of 2 real points separating (in the projective sense) r_0 and $O = p_{\alpha_1}p_{\alpha_2}$;

(ii) *hyperbolic of the* 2*nd kind* if either $P_{\alpha_1\beta}$, $P_{\alpha_2\gamma}$ are conjugate complex, or $P_{\alpha_1\beta}$, $P_{\alpha_2\gamma}$ are real but both $P_{\alpha_1\beta_1}$, $P_{\alpha_1\beta_2}$ and $P_{\alpha_2\gamma_1}$, $P_{\alpha_2\gamma_2}$ are conjugate complex;

(iii) *hyperbolic of the* 1*st kind* in the remaining cases.

29. The various cases alluded to at the beginning of § 28 are all reducible, by making when necessary a suitable change of nomenclature, to the following six:

	Real points	*Conjugate complex points*	
(1)	$P_{11}P_{12}P_{13}P_{21}P_{22}P_{23}P_{31}P_{32}P_{33}$	—	$p_1p_2p_3$, $\pi_1\pi_2\pi_3$ all real,
(2)	$P_{11}P_{12}P_{13}P_{21}P_{22}P_{23}P_{33}$	$P_{31}P_{32}$	
(3)	$P_{13}P_{23}P_{31}P_{32}P_{33}$	$P_{11}P_{12}$, $P_{21}P_{22}$	
(4)	$P_{13}P_{23}P_{33}$	$P_{11}P_{12}$, $P_{21}P_{22}$, $P_{31}P_{32}$	
(5)	$P_{31}P_{32}P_{33}$	$P_{11}P_{21}$, $P_{12}P_{22}$, $P_{13}P_{23}$	p_1p_2, $\pi_1\pi_2$ conjugate complex.
(6)	P_{33}	$P_{11}P_{21}$, $P_{12}P_{22}$, $P_{13}P_{23}$, $P_{31}P_{32}$	

In each of them the graphical representation of section II can, of course, be applied without alteration; but, in order to make clear the distinction between the real elements and the complex ones, we take the precaution of representing a real point P with a spot and two conjugate complex points P with two small loops having the same shape. The drawings corresponding to the above cases, (1)–(6), are to be found in order in

Figs. 1 and 42–6, where, however, only the real lines and the complex lines of the 1st kind of the cubic surface are represented, by a continuous or a dotted line respectively. It is clear from the figures that

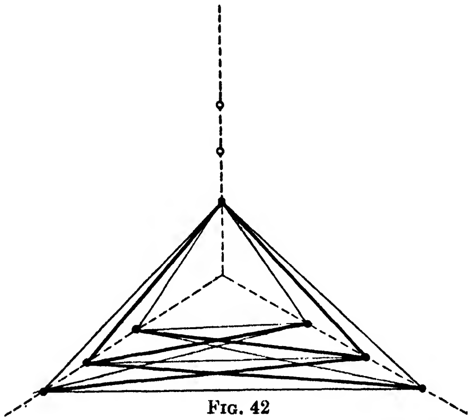


FIG. 42

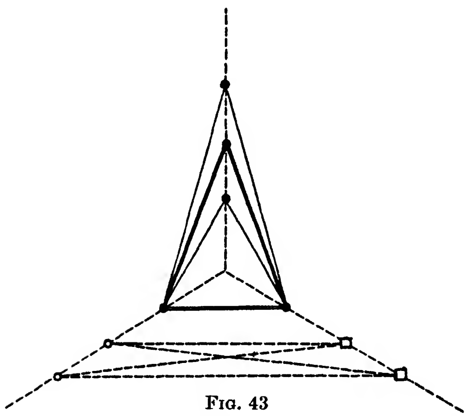


FIG. 43

the numbers of lines of these two types are, respectively, 27 and 0, 15 and 0, 7 and 4, 3 and 12, 3 and 24, 3 and 12; we thus have, taking into account §§ 23 and 24, that:

The non-singular cubic surfaces graphically represented by Figs. 1 *and* 42–6 *are, respectively, of the types* F_1, F_2, F_3, F_4, F_5, F_4. *Conversely, every surface* F_j ($j = 1, 2, \ldots, 5$) *has some real representation of the type* (j),

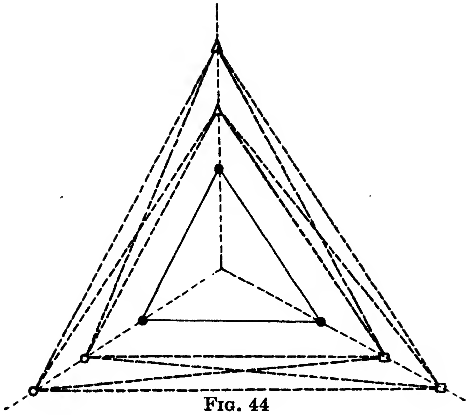


Fig. 44

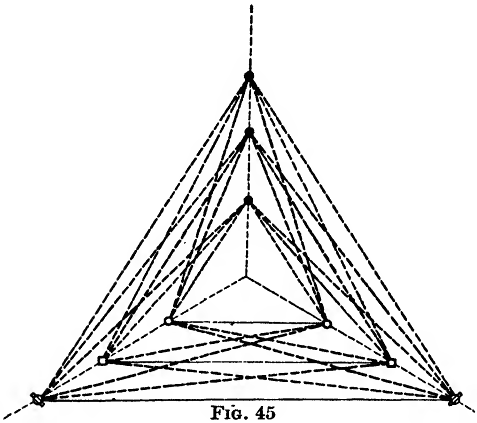


Fig. 45

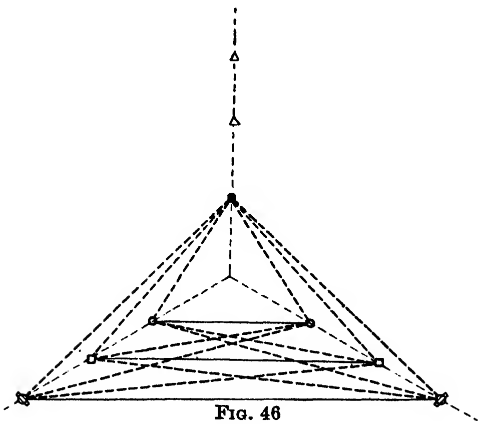


Fig. 46

given by the j*-th of Figs.* 1 *and* 42–5; *and* F_4 *has also some real representation of the type* (6), *given by Fig.* 46.

The determination of all the distinct real graphical representations inherent to any given cubic surface is related to some questions, involving the idea of group, which we discuss in the next section.

30. As an immediate consequence of § 29 we have the following theorem, a suggestive interpretation of which could easily be obtained in [19] (§ 24).

Every (*real non-singular*) *cubic surface of the type* F_1 *or* F_2 *or* F_3 *or* F_4 *can be made to degenerate into* 3 *real planes not belonging to a pencil, and every cubic surface of the type* F_4 *or* F_5 *can be continuously deformed into* 1 *real and* 2 *conjugate complex planes not belonging to a pencil, in such a way that all the intermediate surfaces are real and non-singular* (*and therefore of the same type as the surface initially considered*). *If a real pencil of cubic surfaces contains as a surface a trihedron, upon the edges of which it has* 9 *distinct* (*real or complex*) *base-points, then* all *the surfaces of the pencil sufficiently near to the trihedron are non-singular and of the same type; this type depends only upon the reality of the edges and base-points, in the manner indicated in* § 29.

This theorem both emphasizes one of the features of our graphical representation, and gives easily the means of constructing (and also of studying the shape of) the non-singular cubic surfaces of the various types.

31. From the graphical representations of § 29, taking into due account chapter I and § 28, several remarkable properties of the non-singular cubic surfaces in the real field follow almost at once. We now proceed to enunciate many of them, dealing separately in succession with the five types F_i.

Two conjugate complex lines of the 1st kind of a cubic surface are joined by a real tritangent plane, which we call *of the 3rd kind*; the two lines intersect in a real point, which we call the *principal point* of this tritangent plane.

(i) Surfaces F_1.

A cubic surface F_1 has its 27 lines all real: 12 of them are elliptic, and 15 are hyperbolic of the 1st kind. *The* 12 *elliptic lines of a surface* F_1 *form a double-six*: thus, for instance, the 12 elliptic lines inherent to Fig. 1 are represented in Fig. 47, i.e. constitute the double-six {222}; we say that such a double-six is of the 1st kind.

Fifteen of the other double-sixes are permutable with the former, and we describe them as 'of the 2nd kind'; they contain only 4 elliptic lines, constituting 2 pairs of incident lines without points in common. Each

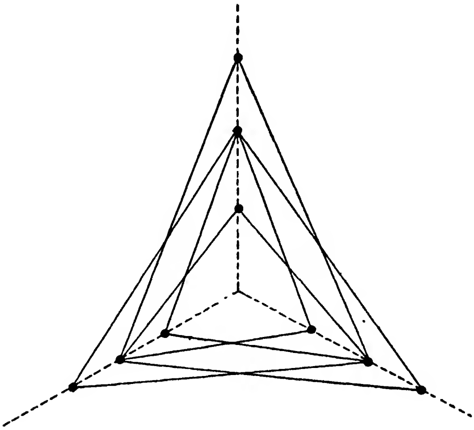


FIG. 47

of the remaining 20 double-sixes, which we call 'of the 3rd kind', contains exactly 6 elliptic lines, constituting 2 complementary triplets.

Also the 45 tritangent planes of F_1 are, of course, all real. Fifteen of them, which we call 'of the 1st kind', join the 15 hyperbolic lines 3 by 3; each of the 30 others, which we call 'of the 2nd kind', contains 1 hyperbolic and 2 elliptic lines of F_1.

(ii) SURFACES F_2.

The 15 real lines of a cubic surface F_2 are of two different sorts: 6 of them, constituting 2 *complementary triplets* (and represented in Fig. 42 with thick strokes), are elliptic; the 9 others are hyperbolic of the 1st kind and constitute a *Steiner set*. The remaining lines of F form a *self-conjugate double-six*, which we call 'of the 9th kind', in which each pair of corresponding lines is a pair of conjugate complex lines of the 2nd kind; hence the σ-transformation inherent to it changes each line of F_2 in its conjugate.

F_2 has 15 other self-conjugate double-sixes, which are the 15 double-sixes permutable with the one just considered. Each sextuplet of such a double-six is consequently self-conjugate, and consists of 4 real lines and 2 conjugate complex lines of the 2nd kind; but, while in 9 of these 15 double-sixes (which we call 'of the 4th kind') the 4 real lines of each

sextuplet are 2 elliptic and 2 hyperbolic, in the 6 others (which we call 'of the 5th kind', and which constitute 2 permutable triads of associate double-sixes) the 4 real lines of each sextuplet are 1 elliptic and 3 hyperbolic. If ω is any one of the 15 double-sixes considered, its 8 real lines are the only real lines of F_2 skew to a well-determined real line of F_2, which is hyperbolic or elliptic according as ω is of the 4th or 5th kind.

It appears, moreover, from Fig. 42 that F_2 has 15 real tritangent planes. Six of them are of the first kind and form a Steiner trihedral pair; the others are of the 2nd kind and can be defined as the 9 planes joining the 6 elliptic lines 2 by 2 in all possible manners.

(iii) SURFACES F_3.

Among the 7 real lines of a cubic surface F_3, exactly one (represented with a double stroke in Fig. 43) is hyperbolic of the 2nd kind; the 6 others belong in pairs to 3 tritangent planes through the former line; one of these pairs (represented with thick strokes in Fig. 43) is of elliptic lines, and the remaining 4 real lines are hyperbolic of the 1st kind. Fig. 43 shows, moreover, that *the* 16 *complex lines of the* 2*nd kind of* F_3 *are the* 16 *lines of the surface which are skew to its hyperbolic line of the* 2*nd kind.*

F_3 has 8 self-conjugate double-sixes. In 2 of these (given by $\{13\}$ and $\{23\}$ in Fig. 43) the 2 sextuplets are mutually conjugate, and both consist of 2 complex lines of the 1st kind and of 4 complex lines of the 2nd kind; these 2 double-sixes are permutable, having in common only the 4 complex lines of the 1st kind of F_3, and we call them 'of the 10th kind': the σ-transformations inherent to them have a product which changes each line of F_3 into its conjugate complex. The remaining self-conjugate double-sixes are exactly the 6 double-sixes permutable with the former 2 (§ 15); each of their sextuplets is self-conjugate, and consists of 2 real lines and 2 pairs of conjugate complex lines of the 2nd kind. We can distribute such double-sixes in 2 sets, one consisting of 2 permutable double-sixes (which we call 'of the 6th kind') and the other consisting of 4 double-sixes (which we call 'of the 7th kind'), for which the 2 real lines of each sextuplet are, respectively, both hyperbolic of the 1st kind, or 1 elliptic and 1 hyperbolic of the 1st kind.

F_3 has only 5 real tritangent planes, which are the tritangent planes through its hyperbolic line of the 2nd kind. We have already considered 3 of them, of which 2 are of the 1st kind and 1 is of the 2nd kind; each of the remaining 2 is of the 3rd kind.

(iv) SURFACES F_4.

The 3 real lines of a cubic surface F_4 are all hyperbolic of the 2nd kind and lie in a plane. Through each of them there are 2 distinct tritangent planes of the 3rd kind, and we shall shortly see that *the* 3 *pairs of principal points of these tritangent planes are the* 3 *pairs of opposite vertices of a plane quadrilateral.*

The 12 complex lines of the 1st kind of F_4 form a 12-*set* (cf. § 10 and Fig. 44), i.e. there are 2 perspective (complex) tetrahedra, whose non-corresponding faces intersect 2 by 2 along the 12 lines. The 4 pairs of corresponding faces are conjugate complex and intersect in 4 real coplanar lines: the vertices of the quadrilateral formed by these lines are the points of intersection of the 6 pairs of conjugate complex lines of the 12-set, which can be more exactly specified as in the preceding statement.

The further lines of F_4 constitute a *double-six*, which we call 'of the 8th kind', each sextuplet of which is self-conjugate and consists of 3 pairs of conjugate complex lines of the 2nd kind. On F_4 there are 3 other self-conjugate double-sixes, which we call 'of the 11th kind', making with the former one a set of 4 mutually permutable double-sixes; their 2 sextuplets are mutually conjugate, and each consists of 4 complex lines of the 1st kind and 2 complex lines of the 2nd kind. The 3 σ-transformations inherent to the 3 double-sixes of the 11th kind have a product which changes each line of F_4 into its conjugate complex.

The only real tritangent planes of F_4 are the 7 already considered: namely, the tritangent plane of the 1st kind which contains the 3 real lines of F_4, and the 3 pairs of tritangent planes of the 3rd kind through these lines.

(v) SURFACES F_5.

A cubic surface F_5, like a surface F_4, has 3 real lines which are all hyperbolic of the 2nd kind and coplanar; in the present case, however, through each real line there are 4 (instead of only 2) tritangent planes of the 3rd kind (Fig. 45). By virtue of § 9, *the principal points of these* 3 *sets of* 4 *tritangent planes are the vertices of* 3 *desmic tetrahedra.*

F_5 contains 12 self-conjugate double-sixes, which we call 'of the 12th kind', and are exactly the 12 double-sixes which, in conformity with a general result of § 15, we can form by suitably choosing 12 of the 24 complex lines of F_5. The 2 sextuplets of such a double-six are mutually conjugate, and each consists of 6 complex lines of the 1st kind.

The 12 σ-transformations inherent to the 12 self-conjugate double-sixes can be distributed in 3 sets of 4 mutually permutable transformations; the 4 σ-transformations of each of the 3 sets have as product the same transformation, which changes each line of F_5 into its conjugate complex.

F_5 has 13 real tritangent planes, which are the tritangent planes through its 3 real lines: one of them (of the 1st kind) is the plane joining these lines, and the 12 others (of the 3rd kind) are the planes touching F_5 at the vertices of the 3 desmic tetrahedra considered above.

32. From § 31 follows that the real tritangent planes of a non-singular cubic surface, F, can only be of one of the following 3 types:

Tritangent planes of the

1st kind, containing 3 hyperbolic lines;

2nd kind, containing 2 elliptic lines and 1 hyperbolic;

3rd kind, containing 1 hyperbolic line and 2 conjugate complex.

Another consequence of § 31 is that the *self-conjugate double-sixes* fall under the following 12 categories:

Kind of the double-six	*Its two sextuplets are*	*Each sextuplet consists of:*			
		Elliptic lines	*Hyperbolic lines of the 1st kind*	*Complex lines of the 1st kind*	*Complex lines of the 2nd kind*
1st	self-conjugate	6	. .	. .	. .
2nd	,,	2	4	. .	. .
3rd	,,	3	3	. .	. .
4th	,,	2	2	. .	2
5th	,,	1	3	. .	2
6th	,,	. .	2	. .	4
7th	,,	1	1	. .	4
8th	,,	. .	. .	. .	6
9th	mutually conj.	. .	. .	. .	6
10th	,,	. .	. .	2	4
11th	,,	. .	. .	4	2
12th	,,	. .	. .	6	. .

In order to avoid misunderstanding, we notice that sextuplets both of the 8th and of the 9th kinds consist of 6 complex lines of the 2nd kind; but, while in the former case the 6 lines can be split up into 3 pairs of conjugate complex lines, this is not possible in the latter case. We also remark that a real hyperbolic line of the 2nd kind of a cubic surface never belongs to a self-conjugate double-six. A direct study and construction of the real double-sixes (namely, of those of the first three kinds) will be found later on, in section X.

Since a 12-set of lines can be derived uniquely from one of the sets

of 15 lines residual to a double-six by suppressing one of its triads of coplanar lines (§ 10), we easily deduce that the *self-conjugate* 12-*sets* fall under the following categories:

Kind of the 12-set	*It consists of:* *Ell. lines*	*Hyp. lines*	*Compl. lines of the 1st kind*	*Compl. lines of the 2nd kind*	*The 2 perspective tetrahedra defined by it being*	*Real faces of each tetrahedron*	*Pairs of conj. compl. corresponding faces*
1st	..	12	..	..	self-conjugate	4	..
2nd	4	8	..	..	,,	4	..
3rd	6	6	..	..	,,	4	..
4th	8	4	..	..	,,	4	..
5th	..	..	4	8	,,	..	..
6th	..	2	10	..	,,	2	..
7th	..	4	..	8	mutually conj.	..	..
8th	2	2	..	8	,,	..	..
9th	..	2	2	8	,,	..	2
10th	2	..	2	8	,,	..	2
11th	..	..	12	..	,,	..	4

More particularly, the numbers and kinds of the self-conjugate 12-sets which can be derived in the above manner from a given self-conjugate double-six, and which we call complementary to it, are given by the following table, which is easily obtained by means of our graphical representations.

Kind of the double-six	*Complementary 12-sets* *Number*	*Kind*	*Total number*
1st	15	1st	15
2nd	12 3	3rd 4th	15
3rd	9 6	2nd 3rd	15
4th	1 2	7th 8th	3
5th	3	8th	3
6th	1 2	5th 10th	3
7th	1 2	5th 9th	3
8th	1 6	11th 6th	7
9th	9 6	2nd 3rd	15
10th	1 2	7th 8th	3
11th	1 2	5th 9th	3
12th	1 6	11th 6th	7

Still using our graphical representation, and taking also into account § 8, it follows easily that the *self-conjugate Steiner sets* fall into 7 categories as follows:

Kind of the Steiner set	*It consists of*				*Remarks*
	Ell. lines	*Hyp. lines*	*Compl. lines of the 1st kind*	*Compl. lines of the 2nd kind*	
1st	6	3	..	..	The 3 hyperbolic lines form a triplet.
2nd	4	5	..	..	The 4 elliptic lines form a skew quadrilateral.
3rd	..	9	..	..	..
4th	3	..	..	6	The 3 real lines form a triplet.
5th	1	2	..	6	The 3 real lines form a triplet.
6th	..	1	4	4	The 4 complex lines of the 2nd kind are skew with the real line and form a skew quadrilateral.
7th	..	3	6	..	The 3 real lines lie in a plane.

We notice that this classification has an *intrinsic* meaning, that is, it refers to the Steiner sets independently of the cubic surfaces to which they belong; through a Steiner set there is, in fact, a pencil of cubic surfaces which all induce the same involution on each line of the set, the involution being determinable on such a line by its intersections with two convenient pairs of lines of the set itself. The first 3 kinds considered above are those which correspond to a Steiner trihedral pair whose 2 trihedra each have 3 real faces, so that they give a projective classification of such pairs of trihedra (and dually, therefore, of 2 real triangles generally situated in the projective space); the 4th and 5th kinds are generated by 2 complex and mutually conjugate Steiner trihedra; the 6th kind corresponds to 2 Steiner trihedra both consisting of 1 real plane and 2 conjugate complex planes; finally the 7th kind is inherent to 2 Steiner trihedra consisting one of 3 real planes and the other of 1 real plane and 2 conjugate complex planes.

For the sake of convenience, we also consider the (non-self-conjugate) Steiner sets consisting of 3 complex lines of the 1st kind (forming a

Cubic surface of the type	*Contains self-conjugate triads of complementary Steiner sets*	
	In number of	*Each consisting of 3 Steiner sets of the following kinds*
F_1	10	1st, 1st, 3rd
	30	2nd, 2nd, 2nd
F_2	1	3rd, 4th, 4th
	9	2nd, 5th, 5th
F_3	4	5th, 5th, 6th
F_4	2	6th, 6th, 6th
	4	7th, 8th, 8th
F_5	16	7th, 9th, 9th

triplet) and 6 complex lines of the 2nd kind, or of 9 complex lines of the 1st kind, and we call them 'of the 8th and 9th kinds' respectively. Then the *self-conjugate triads of complementary Steiner sets* which can be formed with the 27 lines of a non-singular cubic surface of any kind are only of 8 distinct types, as shown by the table at foot of p. 54.

We see, in particular, that each of such triads consists of 3 self-conjugate Steiner sets, with the exception of those of the last two types which contain 1 self-conjugate and 2 mutually conjugate Steiner sets.

33. Many of the results of §§ 31, 32 are summarized and compared in the following table, in which the suffixes refer to the kinds, as specified above.†

Has \ *A real non-singular cubic surface of the type*			F_1	F_2	F_3	F_4	F_5
Lines	real	elliptic	12	6	2	..	..
		hyp. of the 1st kind	15	9	4	..	..
		hyp. of the 2nd kind	..	..	1	3	3
	complex	of the 1st kind	..	12	4	12	24
		of the 2nd kind	..	..	16	12	..
Real tritangent planes		of the 1st kind	15	6	2	1	1
		of the 2nd kind	30	9	1	..	..
		of the 3rd kind	..	..	2	6	12
Self-conjugate double-sixes whose 2 sextuplets are		self-conjugate	1_1	9_4	2_6	1_8	..
			15_2	6_5	4_7	..	..
			20_3	..	..	..	..
		mutually conjugate	..	1_9	2_{10}	3_{11}	12_{12}
Self-conjugate 12-sets originated by 2 homologous tetrahedra which are		self-conjugate	15_1	9_2	6_5	3_5	72_6
			180_2	6_3	..	6_6	..
			300_3	..	..	..	..
			45_4	..	..	..	..
		mutually conjugate	..	9_7	2_7	6_9	12_{11}
			..	36_8	4_8	1_{11}	..
			..	..	8_9	..	..
			..	..	4_{10}	..	..
Self-conjugate Steiner sets originated by 2 Steiner trihedra which are		self-conjugate	20_1	9_2	4_6	6_6	16_7
			90_2	1_3	..	4_7	..
			10_3	..	..	..	..
		mutually conjugate	..	2_4	8_5	..	..
			..	18_5	..	..	..

This table shows up several interesting facts; and we shall now point out two of their most obvious consequences. Firstly we notice that the

† From it we see, for instance, that a cubic surface F_1 contains 36 self-conjugate double-sixes, of which 1 is of the 1st kind, 15 are of the 2nd kind, and 20 are of the 3rd kind, etc.

surfaces F_5 are the only non-singular cubic surfaces which have no self-conjugate sextuplets, so that *they have no real birational representation upon a plane* (in accordance with the fact, which we shall prove later on, in § 63, that they consist of 2 separate pieces): and this, by the way, is one of the less obvious reasons for the superiority of our graphical representation—whose validity in the real field has no exceptions—to Cremona's plane representation. We remark, secondly, that if a Steiner set of the 1st or of the 4th kind belongs to a non-singular cubic surface, this can only be a surface F_1 or F_2 respectively; the 9 lines of such a set, therefore, constitute the base curve of a pencil of cubic surfaces which, if (real and) non-singular, are all of the same type: it follows, by virtue of § 25, that *in such a pencil (which can be determined by 2 real or conjugate complex trihedra) all the real surfaces are non-singular and of the same type, F_1 or F_2 respectively, with the solitary exception of the 2 real trihedra in the first case.*

We can, of course, make several choices among the characters considered above for a real cubic surface if we only wish to make a distinction among the various types F_1, F_2, F_3, F_4, F_5; these can, for instance, be characterized by means of the corresponding number of self-conjugate Steiner sets, which is, respectively, 120, 30, 12, 10, 16. Another choice, which is that considered by previous authors,† is given by the following table:

Cubic surfaces of the type / *Number of their*	F_1	F_2	F_3	F_4	F_5
Real lines	27	15	7	3	3
Real tritangent planes. .	45	15	5	7	13

IX. The group of the lines of the real cubic surfaces

34. In correspondence with all the circulations *in the real domain* of a (real) non-singular cubic surface F, which keep it non-singular, we obtain a well-defined group Γ of substitutions among the 27 (real or complex) lines of F. Γ is a sub-group of $\mathfrak{G}$ which is *intransitive*, since its substitutions must transform each line of F into a line of the same

† Most of them are not concerned with considerations of continuity like those here developed in section VII, so that the 5 types have been sometimes introduced in a different order; for instance, in L. Schläfli, 'On the Distribution of Surfaces of the Third Order into Species . . .', *Phil. Trans. R. Society*, vol. 153 (1864), pp. 193–241, § 6, and in J. A. Todd, 'On questions of reality for certain geometrical loci', *Proc. London Math. Soc.* (II), vol. 32 (1930), pp. 449–87, § 4, our 4th and 5th types are interchanged.

type (real, elliptic, or hyperbolic of the 1st or 2nd kind, or complex of the 1st or 2nd kind); Γ is not, however, the totality of the substitutions of $\mathfrak{G}$ which satisfy this condition; this—in the case in which F contains some elliptic line—can be seen *a priori* as follows.

If r is an elliptic line of F, each direction of movement of a point upon it determines a well-defined direction of rotation of its tangent plane around r (§ 27): and, with the usual meaning of the word, we can indicate which of the two distinct possibilities thus arises, by saying that r is *right-* or *left-handed* with respect to F. During every continuous variation of F in [3] which preserves its non-singularity, any elliptic line of F keeps its right- or left-handed character.

On a cubic surface F', transform of F by a *negative homography*,† the line r' corresponding to r (which is still elliptic) has opposite character. But (§ 24) it is certainly possible to perform a continuous deformation of F' into F without any intermediate position being singular, so that F and F' have the same number both of right- and of left-handed elliptic lines, and we can say that:

The elliptic lines of F are distributed in 2 sets, one of right-handed and the other of left-handed lines, which are non-equivalent with respect to the transformations of Γ; the 2 sets contain the same number of lines.

We now proceed to the exact determination of Γ, by studying separately the 5 different cases that F can present (§ 23), and we denote by Γ_i the group Γ inherent to a cubic surface of the type F_i.‡ We can in any case let F degenerate into a trihedron $F_0 = \pi_1 \pi_2 \pi_3$ as in § 28, and suppose the 27 (real or complex) lines of F represented symbolically as in § 4 and graphically as in § 29.

(i) THE GROUP Γ_1 OF THE SURFACES F_1

35. If r denotes one of the 12 elliptic lines of a surface F_1, we orientate it arbitrarily, namely, we choose on it one direction of movement and around it the corresponding direction of rotation (§ 27), and consider

† Namely, by a real non-singular homography of [3] into itself, which (like a symmetry with respect to a plane) can be represented analytically by a linear substitution upon the (real) homogeneous projective coordinates having negative determinant. It is obvious what we mean by speaking of a *positive homography*, and that the necessary and sufficient condition that a given homography be continuously reducible to the identity (remaining in the domain of the real non-singular homographies) is that it is positive.

‡ The attack on this problem which follows could be simplified in some details by considering particular cubic surfaces having convenient groups of projective transformations into themselves (surfaces to which section XIV is dedicated); we prefer, however, a direct approach to the question, which is interesting in itself and more suitable for further deductions.

through it the 5 tritangent planes $\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5$ of F_1, which we suppose occurring in their oriented pencil in this order (defined, of course, save for a circular substitution). The plane α_i ($i = 1, 2, \dots, 5$) contains further one elliptic line and one hyperbolic line of F_1 (§ 32), intersecting r respectively in A_i, A'_i; these 2 points correspond in the elliptic involution defined on r by F_1 (§ 27), so that they are distinct. The 5 pairs $(A_i A'_i)$ of this involution correspond projectively to the 5 planes α_i, and are therefore distinct; the 10 distinct points $A_1 A_2 A_3 A_4 A_5 A'_1 A'_2 A'_3 A'_4 A'_5$ occur, moreover, upon r in such a cyclic order that both A_i and A'_i can only be followed by A_{i+1} or A'_{i+1} (with $A_6 = A_1$, $A'_6 = A'_1$): we shall now prove that *their cyclic order is exactly* $A_1 A'_2 A_3 A'_4 A_5 A'_1 A_2 A'_3 A_4 A'_5$.

Let us suppose r to be, for instance, the line 120 (Fig. 48); then, by conveniently orientating this line, we can take either

$$\alpha_1 = (121),\quad \alpha_2 = (122),\quad \alpha_3 = (123),\quad \alpha_4 = \begin{pmatrix}120\\210\\330\end{pmatrix},\quad \alpha_5 = \begin{pmatrix}120\\230\\310\end{pmatrix},$$

or

$$\alpha_1 = (121),\quad \alpha_2 = (122),\quad \alpha_3 = (123),\quad \alpha_4 = \begin{pmatrix}120\\230\\310\end{pmatrix},\quad \alpha_5 = \begin{pmatrix}120\\210\\330\end{pmatrix},$$

since in both cases, when $F_1 \to F_0$, the 5 planes α_1, α_2, α_3, α_4, α_5 tend respectively to the planes $r_0 P_{31}$, $r_0 P_{32}$, $r_0 P_{33}$, π_3, π_3 which occur in this cyclic order in the pencil of axis $r_0 = P_{11} P_{22}$. The points P_{11} and P_{22}, as is clearly shown by Fig. 48, are therefore the limiting positions of the triads of points $A'_1 A_2 A'_3$ and $A_1 A'_2 A_3$, so that, on the orientated line r, A_1 can but be followed by A'_2, etc. If the orientation which we obtain at the limit on r_0 is, for instance, that marked in Fig. 48 on the representative line $\bar{r}_0$, the first of the two above alternatives holds, and the limiting positions of $A_4 A_5 A'_4 A'_5$ are respectively those represented in Fig. 48 by $\bar{A}_4 \bar{A}_5 \bar{A}'_4 \bar{A}'_5$; which completes the proof of the theorem.

36. Let us now consider one of the 15 hyperbolic lines of F_1, which we still denote by r, and fix arbitrarily an orientation in the pencil of axis r. The 5 tritangent planes α_i through r are all contained in the complete angle $\lambda\mu$ of this pencil which is filled by the planes touching F_1 along r, λ and μ being the parabolic planes through r (§ 27), and occur within this angle in a well-defined order which we can suppose to be $\lambda\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5\mu$. We shall prove that, in every case, α_1, α_3, α_5 *are*

of the 1*st kind* (namely, each of them further contains 2 hyperbolic lines) *and* α_2, α_4 *are of the* 2*nd kind* (that is, each of them further contains 2 elliptic lines).

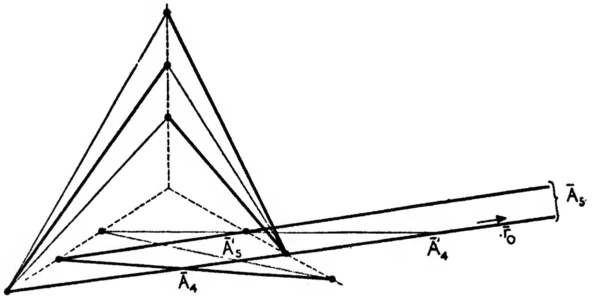


Fig. 48

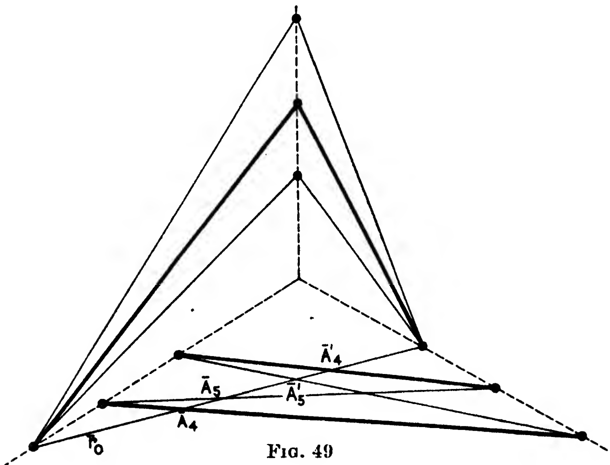


Fig. 49

For this purpose we can, for instance, suppose r to be the line 130 (Fig. 49), A_i, A'_i still having the same meaning as in § 35 (save that they now play a symmetric role); we have that, on r, the 5 pairs of points $A_1A'_1$, $A_2A'_2$, $A_3A'_3$, $A_4A'_4$, $A_5A'_5$ belong to a hyperbolic involution and occur in this order when a pair of corresponding points describes once and continuously the whole involution from the double point which corresponds to λ to the double point which corresponds to μ. When $F_1 \to F_0$, 3 of those 5 pairs of points tend to $P_{11}P_{23}$, and the other 2 tend to the pairs of points represented in Fig. 49 by $\bar{A}_4\bar{A}'_4$ and

$\bar{A}_5 \bar{A}_5'$, so that (by conveniently choosing the orientation around r) we can only have either

$$\alpha_1 = (131), \quad \alpha_2 = (132), \quad \alpha_3 = (133), \quad \alpha_4 = \begin{pmatrix}130\\210\\320\end{pmatrix}, \quad \alpha_5 = \begin{pmatrix}130\\220\\310\end{pmatrix},$$

or

$$\alpha_1 = (133), \quad \alpha_2 = (132), \quad \alpha_3 = (131), \quad \alpha_4 = \begin{pmatrix}130\\210\\320\end{pmatrix}, \quad \alpha_5 = \begin{pmatrix}130\\220\\310\end{pmatrix},$$

and our theorem is obvious from Fig. 49.

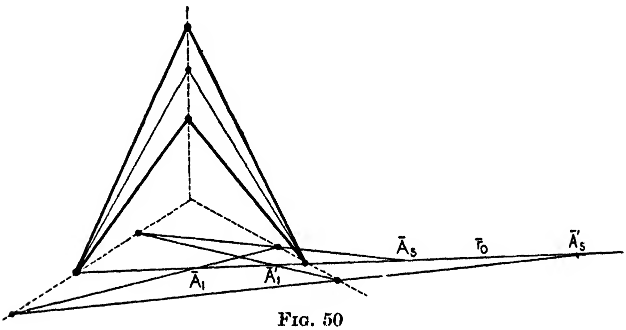


FIG. 50

Inverting the orientation in the pencil of axis r has merely the effect of interchanging λ and μ, α_1 and α_5, α_2 and α_4, leaving α_3 unaltered; it follows that α_3 is a tritangent plane of the 1st kind which is intrinsically related to the hyperbolic line r: we call it the *principal plane* inherent to this line.

According as the former or the latter of the above two alternatives holds, the principal plane inherent to $r = 130$ is either (133) or (131); but Fig. 50 shows that the principal plane inherent to the line 220 is uniquely determined as the plane (222), which is therefore inherent to each of its 3 lines. As we shall see later (§ 38) that Γ_1 operates transitively upon the 15 hyperbolic lines of F_1, it follows that:

The principal planes of F_1 are only 5 *in number; they contain all the* 15 *hyperbolic lines of F, and each of them is inherent to each of its* 3 *lines.*

Also taking into account § 10, we see that the 5 principal planes of the cubic surface F_1 represented by Fig. 1 can be either

(222), (111), (133), (313), (331)

or (222), (333), (311), (131), (113).

The former or the latter case actually occurs, according to the manner in which F_1 degenerates into F_0.

A non-principal tritangent plane of the 1st kind and a tritangent plane of the 2nd kind through r are to be called *non-consecutive* or *consecutive* respectively according to whether or not they are separated (in the projective sense) by the principal tritangent plane and the remaining tritangent plane of the 1st kind through r; thus, for instance, α_1, α_2 are consecutive and α_1, α_4 are non-consecutive.

37. If α is a tritangent plane containing 3 lines of F_1 which have a point D in common, each of these lines is clearly hyperbolic, so that α is of the 1st kind. Upon any one, r, of the 3 lines of α, for which we keep the notation of § 36, D is one of the double points of the involution defined by F_1; the tritangent plane α coincides in this case with one of the two parabolic planes λ, μ, so that it can only be one of the planes α_1, α_5. Hence:

A tritangent plane of F_1 which contains 3 lines of a pencil is necessarily one of the 10 non-principal tritangent planes of the 1st kind.

38. We are now in a position to prove that:

The group Γ_1 inherent to a cubic surface F_1 is icosahedral, and can be defined as the group of the 60 substitutions of $\mathfrak{G}$ which transform into themselves each of the 2 sextuplets of elliptic lines and also the system of 5 principal tritangent planes of F_1, inducing among these planes a substitution of even class.

The 12 elliptic lines of F_1 constitute a double-six δ (§ 31), for which we adopt the Schläfli notation of § 22. First of all we have that, in every case, 1 *of the* 2 *sextuplets of* δ *consists of right-handed and the other of left-handed lines.* Two triplets like $a_1 a_2 a_3$ and $b_4 b_5 b_6$ are, in fact, complementary, i.e. they belong to a single quadric Q; the planes touching Q at the points $a_1 b_4$, $a_1 b_5$, $a_1 b_6$ are respectively the planes $a_1 b_4$, $a_1 b_5$, $a_1 b_6$, so that a_1 (and likewise each of the lines $a_2 a_3 b_4 b_5 b_6$) is right-handed for F_1 if, and only if, it is right-handed for Q. But on every ordinary ruled quadric Q all the generators of a system are right-handed and all the generators of the other system are left-handed, since generators of the same system or of opposite systems are transitively interchanged by the homographic transformations of Q into itself which respectively are positive or negative; it follows that all the lines of one of the 2 complementary triplets are right-handed and all the lines of the other triplet are left-handed (for Q and therefore also for F_1): whence the result stated.

Let us choose any one of the 12 lines of δ, for instance b_6, and fix arbitrarily a direction of rotation around it; then we can suppose without restriction that the 5 planes

$$\alpha_1 = a_1 b_6, \quad \alpha_2 = a_2 b_6, \quad \alpha_3 = a_3 b_6, \quad \alpha_4 = a_4 b_6, \quad \alpha_5 = a_5 b_6$$

occur in their oriented pencil in this cyclic order, so that, by virtue of § 35, the 5 points $A_i = a_i b_6$ occur upon the line b_6—oriented concordantly—in the cyclic order $A_1 A_3 A_5 A_2 A_4$.

We now take arbitrarily in an oriented pencil, of axis b_6', 5 distinct planes $\alpha_1' \alpha_2' \alpha_3' \alpha_4' \alpha_5'$ occurring in this cyclic order; and, upon the line b_6', oriented at will, 5 distinct points $A_1' A_3' A_5' A_2' A_4'$ occurring in this cyclic order. If i_1, i_2, i_3, i_4 are any 4 distinct values among the numbers 1, 2, 3, 4, 5, the cross-ratios $(A_{i_1}' A_{i_2}' A_{i_3}' A_{i_4}')$ and $(\alpha_{i_1}' \alpha_{i_2}' \alpha_{i_3}' \alpha_{i_4}')$ are (finite, non-nul, and) opposite in sign, so that the conditions (i) of § 21 are all verified. By choosing in α_i' and through A_i' a line a_i' ($i = 1, 2, \ldots, 5$) in such a generic manner that the conditions (ii) of § 21 are also satisfied, we have that the 6 lines $a_1' a_2' a_3' a_4' a_5' b_6'$ belong to a double-six

$$\delta' = \begin{pmatrix} a_1' a_2' a_3' a_4' a_5' a_6' \\ b_1' b_2' b_3' b_4' b_5' b_6' \end{pmatrix},$$

lying on a well-defined non-singular cubic surface F' (§ 21). We shall prove that, if b_6 and b_6' are both right- or left-handed, it is possible to deform $a_1 a_2 a_3 a_4 a_5 b_6$ continuously into $a_1' a_2' a_3' a_4' a_5' b_6'$ respectively (remaining in the real field), in such a way that at any stage of the deformation the conditions (i), (ii) of § 21 still hold.

It is evidently possible to deform continuously the figure

$$(b_6 A_1 A_2 A_3 A_4 A_5 \alpha_1 \alpha_2 \alpha_3 \alpha_4 \alpha_5)$$

into the figure $\quad (b_6' A_1' A_2' A_3' A_4' A_5' \alpha_1' \alpha_2' \alpha_3' \alpha_4' \alpha_5'),$

so that for every intermediate position $(\bar{b}_6 \bar{A}_1 \bar{A}_2 \bar{A}_3 \bar{A}_4 \bar{A}_5 \bar{\alpha}_1 \bar{\alpha}_2 \bar{\alpha}_3 \bar{\alpha}_4 \bar{\alpha}_5)$ the conditions (i) hold. We can at once extend such a deformation to the 4 lines a_1, a_2, a_3, a_4 by choosing arbitrarily—in every intermediate stage—$\bar{a}_j$ distinct from $\bar{b}_6$ in the pencil of centre $\bar{A}_j$ and plane $\bar{\alpha}_j$ ($j = 1, 2, 3, 4$). Since the cross-ratios $(\bar{A}_1 \bar{A}_2 \bar{A}_3 \bar{A}_4)$ and $(\bar{\alpha}_1 \bar{\alpha}_2 \bar{\alpha}_3 \bar{\alpha}_4)$ are different, the 4 lines $\bar{a}_1 \bar{a}_2 \bar{a}_3 \bar{a}_4$ have a further common transversal $\bar{b}_5$, skew to $\bar{b}_6$: denoting by $\bar{L}$ the point of intersection of $\bar{b}_5$ and $\bar{\alpha}_5$, in order to satisfy completely (i) and (ii) we have to choose $\bar{a}_5$—in the pencil of centre $\bar{A}_5$ and plane $\bar{\alpha}_5$—in any position distinct from $\bar{b}_6$ and $\bar{l} = \bar{A}_5 \bar{L}$. Hence we can arrange our deformation in such a way that $\bar{a}_j \to a_j'$ ($j = 1, 2, 3, 4$); but, if a_5^*, b_5', and l' are respectively the limiting positions of $\bar{a}_5$, $\bar{b}_5$, and $\bar{l}$, it is *a priori* possible that a_5^* may not

coincide with a_5'. However, if the 2 lines a_5^*, a_5' belong to the same angle $b_6'l'$, it is manifestly possible to deform $a_1'a_2'a_3'a_4'a_5^*b_6'$ in the required manner into $a_1'a_2'a_3'a_4'a_5'b_6'$, simply by conveniently rotating a_5^* in the pencil of centre A_5' and plane α_5'. If, on the contrary, a_5^* and a_5' separate b_6' and l', we can reduce to the case already considered by applying to $a_1'a_2'a_3'a_4'a_5^*b_6'$ the harmonic biaxial homography of axes b_5' and b_6'; this homology, in fact, is a *positive* homography, which transforms each of the lines $a_1'a_2'a_3'a_4'b_6'$ into itself, and a_5^* into a line belonging to the angle $b_6'l'$ which contains a_5'.

If we take into account §§ 21, 24, 31, we have in conclusion that F' is a cubic surface of the 1st type, the 12 elliptic lines of which make up δ'; in other words, the double-six δ'—of which we gave above the *construction*—is the most general double-six of the 1st kind. Moreover, if δ and δ' are any 2 double-sixes of the 1st kind, of which $a_i a_6$ and $a_{i'}' a_6'$ are any 2 pairs of distinct lines, all either right- or left-handed, we obtain—in the manner explained—exactly two essentially distinct continuous deformations of δ into δ' which transform a_i into $a_{i'}'$ and b_6 into b_6' (and therefore also a_6 into a_6'), since, for instance, b_6' can be orientated in two distinct manners. In particular, if δ' coincides with δ, namely, if F' coincides with F, we have exactly 2 distinct operations of Γ_1 which transform, for instance, a_5 and a_6 into any 2 distinct lines of the set $a_1 a_2 a_3 a_4 a_5 a_6$; the order of Γ_1 is therefore $2.6.5 = 60$. The 15-set residual to δ contains only one line c_{56} incident to a_5 and a_6, and only one line c_{ij} incident to a_i and a_j $(i \neq j)$; in Γ_1 there are 4 distinct operations transforming the pair $(a_5 a_6)$ into the pair $(a_i a_j)$, and therefore also c_{56} into c_{ij}: we see incidentally that, as stated in § 36, all the hyperbolic lines of F_1 are equivalent with respect to Γ_1.

Finally, remembering also §§ 14 and 36, we have that Γ_1 induces a group of substitutions among the 5 principal tritangent planes of F_1, whose order is still 60; this induced group is the alternating group (which is simply isomorphic with the icosahedral group†), since the alternating group is the only subgroup of index 2 contained in a symmetric group.‡

Many other properties connected with the consideration of the group Γ_1 will be discussed later on, in section X.

39. Fig. 47 shows that the 12 elliptic lines of F_1 are distributed into 3 sets of 4 lines of the type

$$a_1 a_2 b_3 b_4, \qquad a_3 a_4 b_5 b_6, \qquad a_5 a_6 b_1 b_2,$$

† Cf., for instance, L. Bianchi, *Lezioni sulla teoria dei gruppi di sostituzioni e delle equazioni algebriche secondo Galois* (Pisa, Spoerri, 1900), chap. iv, § 55.

‡ Cf. L. Bianchi, op. cit., chap. i, § 8.

respectively belonging to the 3 Steiner sets of the fundamental triad T inherent to the plane representation of F_1 (§ 21). An operation of Γ_1 transforming T into itself must therefore transform each of those 3 sets of 4 lines into one of the same sets, and consequently the pair $(a_1 a_2)$ into $(a_1 a_2)$ or $(a_3 a_4)$ or $(a_5 a_6)$. By virtue of § 38 there are in Γ_1 $3.4 = 12$ operations satisfying the last condition; we shall see that

Each of these 12 *operations of* Γ_1 *transforms both* T *and the principal plane* (222) *into themselves.*

For this purpose we notice, from the graphical representation, that the 12 hyperbolic lines of F_1 which do not belong to the tritangent plane (222) can be distributed into 3 sets of 4 lines incident with the same line of this plane; and that F_1 contains 2 (and only 2) distinct triads of complementary Steiner sets, inducing in these 12 lines the above-mentioned distribution into 3 sets of 4 lines: one of them is the fundamental triad T, and the other is the triad T', transform of T by the σ-transformation [222]. The 6 transformations of $\mathfrak{G}$

$$[12]\,[211]\,[233], \qquad [22]\,[121]\,[323], \qquad [32]\,[112]\,[332],$$
$$[12]\,[213]\,[231], \qquad [22]\,[123]\,[321], \qquad [32]\,[132]\,[312],$$

each of which is the product of 3 mutually permutable σ-transformations, transform into themselves each of the 2 sextuplets of δ and the principal plane (222), interchange T and T', and perform the 6 existing distinct transpositions among the other 4 principal planes. Taking also into account § 38, we have consequently in Γ_1 12 operations leaving (222) unaltered, each of which can be expressed as the product of an *even* number of the 6 transformations considered above, so that it transforms T into itself.

Thus our previous statements are proved. Moreover, we have that the distinct triads of complementary Steiner sets which are transforms of T by the 60 operations of the group Γ_1 are in number $60:12 = 5$, each connected with one of the 5 principal planes in the same manner as T is connected with (222); and we can say that:

In the real domain, a cubic surface F_1 *can degenerate into* 3 *independent real planes only in* 5 *essentially distinct ways, leading to* 5 *essentially distinct graphical representations for its* 27 *lines.*†

† This conclusion (and likewise those of §§ 43, 47, 51, 55) should be compared with the final result of § 21, and can easily be interpreted in connexion with the representation of the real cubic surfaces of [3] with the points of [19] (§ 24).

(ii) The group Γ_2 of the surfaces F_2

40. In order to avoid an interruption later on in our deductions, we explain beforehand some results concerning 5 generic points upon a sphere. We say that 5 (real) points of a sphere are *unrelated* if no 4 of them are coplanar; then the plane of any 3 of them leaves the other 2 either on one side or on opposite sides, and we say respectively that the latter 2 points are either *non-separated* or *separated* by the former 3 points. We shall prove that:

5 *unrelated points of a sphere can be distributed* (*in a single way*) *into two sub-sets—the first of* 3 *points and the second of* 2 *points—such that any* 2 *of the* 5 *points are separated or non-separated by the remaining* 3, *according as they belong to the same sub-set or to different sub-sets.*

To see this, we remark that—on the sphere—4 of the 5 given points, $A_1 A_2 A_3 A_4$ say, are joined 3 by 3 by 4 circles dividing the sphere into 10 cells, of which 4 are spherical triangles and 6 are spherical lunes; and the fifth point A_5 is interior to one of these cells. It is then easily seen that, if A_5 is, for instance, inside the spherical triangle $A_1 A_2 A_3$,† the above stated result holds, the 2 sub-sets in which the 5 points distribute being $A_1 A_2 A_3$ and $A_4 A_5$; and the same is true if A_5 is, for instance, inside the spherical lune $A_1 A_2$,‡ the 2 sub-sets being now $A_3 A_4 A_5$ and $A_1 A_2$.

We have, moreover, that, if we consider on a sphere any 2 sets of 5 unrelated points, *it is always possible to deform continuously one set into the other, so that every intermediate position also consists of* 5 *unrelated points.* The correspondence which we thus obtain among the points of the 2 sets, necessarily associates to a point of the first or second sub-set respectively a point of the first or second sub-set; and, by performing all the possible deformations, we obtain only 6 different correspondences, inducing the 6 distinct substitutions among the first sub-sets of the 2 sets. In particular, a set of 5 unrelated points of a sphere has a *group* of 6 deformations into itself, which is simply isomorphic with a symmetric group of degree 3; its operations induce the 6 distinct substitutions among the 3 points of the first sub-set, and each of them interchanges or does not interchange the 2 remaining points, according

† Namely, the spherical triangle determined by the arcs of the circles $A_1 A_2 A_4$, $A_2 A_3 A_4$, $A_3 A_1 A_4$ which respectively have as extremes the points $A_1 A_2$, $A_2 A_3$, $A_3 A_1$ and do not contain the point A_4.

‡ Namely, the spherical lune determined by the arcs of the circles $A_1 A_2 A_3$, $A_1 A_2 A_4$ which have as extremes the points $A_1 A_2$ and do not contain the point A_3 or A_4.

as the substitution induced by it on the first sub-set is of odd or even class.†

By virtue of the Riemannian representation of the points of a complex line by the (real) points of a sphere, the above results can be applied to the sets of 5 *unrelated points of a complex line*, namely, to the sets of 5 distinct points (or complex numbers) no 4 of which have a real cross-ratio. Thus, for instance, we have that:

5 complex unrelated numbers can be distributed (in a single manner) into 2 sub-sets of 3 and 2 numbers, such that 2 of those 5 numbers determine with the other 3, taken in any order, cross-ratios whose imaginary parts are of opposite or the same sign, according as the 2 numbers considered belong to the same sub-set or to different sub-sets.

41. Let us now consider a cubic surface F_2. From its graphical representation (§ 29) we have immediately that 60 pairs of skew lines can be formed with its 15 real lines; of the 5 lines of F incident to such a pair, 2 are conjugate complex (of the 2nd kind) and 3 are real: the latter constitute a triplet whose complementary again contains 3 real lines, 2 of which are the lines of the pair initially considered. Hence we obtain (in accordance with § 8) that $60/6 = 10$ pairs of complementary triplets can be formed with the 15 real lines of F_2. From Fig. 42 it follows, moreover, at once that in 9 of these pairs, which we describe as *of the* 1*st kind*, each triplet contains 1 elliptic and 2 hyperbolic lines; while all the lines of the remaining pair, which we describe as *of the* 2*nd kind*, are elliptic. By virtue of § 34 we have that, in the pair of complementary triplets of the 2nd kind, and consequently also in any pair of the 1st kind, the elliptic lines of one triplet are left-handed and the elliptic lines of the other triplet are right-handed.

Conversely, we can define the most general cubic surface F_2 in the following way. We consider 2 complementary triplets of real lines, namely, 2 triplets of lines $a_1 a_2 a_3$, $b_1 b_2 b_3$, generators of opposite systems of a quadric Q, such that a_r and b_s intersect in a point L_{rs} ($r, s = 1, 2, 3$); let us, moreover, choose 2 distinct lines m, $\bar{m}$, conjugate complex of the 2nd kind, respectively intersecting a_1 and a_2 in the pair of conjugate complex points M_1, $\bar{M}_1$ and M_2, $\bar{M}_2$. The 8 lines

† In the case in which the 5 points are the vertices of a regular triangular double-pyramid, the corresponding group is the group of the movements of the double-pyramid into itself.

$a_1 a_2 a_3 b_1 b_2 b_3 m \bar{m}$ present to the cubic surfaces 19 conditions, which are independent if

$$(L_{11} L_{12} L_{13} M_1) \neq (L_{21} L_{22} L_{23} M_2),$$
whence
$$(L_{11} L_{12} L_{13} \bar{M}_1) \neq (L_{21} L_{22} L_{23} \bar{M}_2), \tag{1}$$

namely, if Q—as we suppose—contains neither m nor $\bar{m}$: in fact, on the contrary hypothesis, we should have (at least) one cubic surface containing these 8 lines and having Q as a component, which is impossible since m and $\bar{m}$ are skew. We therefore have a well-defined cubic surface, F say, through $a_1 a_2 a_3 b_1 b_2 b_3 m \bar{m}$. The surface F is obviously real, and—if reducible—can only reduce to one plane $a_3 b_r$ and to a quadric through $b_s b_t m \bar{m}$ (where $r\ s\ t$ is a permutation of the numbers 1 2 3), which occurs if, and only if,

$$(L_{1s} L_{1t} M_1 \bar{M}_1) = (L_{2s} L_{2t} M_2 \bar{M}_2). \tag{2}$$

Let us suppose that *none of these relations holds*, so that F is irreducible; then, by virtue of a remark already made in § 21, the surface F is non-singular, since it is non-ruled and contains 5 skew lines ($b_1 b_2 b_3 m \bar{m}$) having 2 distinct common transversals (a_1 and a_2). This surface must be of the type F_2 by the results of § 31, and its 27 lines are rationally defined by the 8 lines $a_1 a_2 a_3 b_1 b_2 b_3 m \bar{m}$: in fact, *all the remaining* 19 *lines of F_2 can be constructed linearly*, as the intersections, distinct from these 8 lines, of F_2 with the 9 planes $a_r b_s$, with the 4 planes $a_1 m$, $a_2 m$, $a_1 \bar{m}$, $a_2 \bar{m}$ and with the 6 quadrics containing one of the lines m, $\bar{m}$ and two of the lines b_1, b_2, b_3.

If we consider on a_1 the points M, $\bar{M}$ such that

$$(L_{11} L_{12} L_{13} M) = (L_{21} L_{22} L_{23} M_2), \quad (L_{11} L_{12} L_{13} \bar{M}) = (L_{21} L_{22} L_{23} \bar{M}_2),$$

we see that they are conjugate complex, and distinct from M_1, $\bar{M}_1$ by virtue of (1). Then the relations (2) are equivalent to

$$(L_{1s} L_{1t} M_1 \bar{M}_1) = (L_{1s} L_{1t} M \bar{M}),$$
whence
$$(L_{1s} L_{1t} M M_1) = (L_{1s} L_{1t} \bar{M} \bar{M}_1),$$

so that our hypothesis can be expressed simply by saying that—on a_1—*the 5 points $L_{11} L_{12} L_{13} M M_1$ must be unrelated.*

42. Let us now consider another surface F_2', defined by 8 lines $a_1' a_2' a_3' b_1' b_2' b_3' m' \bar{m}'$ satisfying conditions similar to those given in § 41 for F_2. We ask if it is possible to deform F_2 continuously into F_2', keeping it non-singular, in such a way that a_1 goes into a_1', a_2 goes into a_2', etc. A first condition necessary for this purpose is that, if on Q the lines $a_1 a_2 a_3$ are, for instance, left-handed (and therefore $b_1 b_2 b_3$ are

right-handed), on Q' the lines $a_1' a_2' a_3'$ must be left-handed (so that $b_1' b_2' b_3'$ are right-handed). This condition being satisfied, it is possible to deform Q into Q' (for instance, by means of a positive homography) in such a way that $a_1 a_2 a_3 b_1 b_2 b_3$ go into $a_1' a_2' a_3' b_1' b_2' b_3'$ respectively.

In order to complete the required deformation of F_2 into F_2', we have only to deform at the same time the ordered set $L_{11} L_{12} L_{13} M M_1$ into $L_{11}' L_{12}' L_{13}' M' M_1'$, in such a way that every intermediate set consists of 5 unrelated points; to obtain this, it is necessary that the correspondence among the 2 sets $L_{11} L_{12} L_{13} M M_1$, $L_{11}' L_{12}' L_{13}' M' M_1'$ should associate the first and second sub-set of the former to the first and second sub-set of the latter (§ 40), in which case $L_{11} L_{12} L_{13} M M_1$ can be deformed in the required manner either into $L_{11}' L_{12}' L_{13}' M' M_1'$ or into $L_{11}' L_{12}' L_{13}' \bar{M}' \bar{M}_1'$, so that m, $\bar{m}$ go either into m', $\bar{m}'$ or into $\bar{m}'$, m' respectively.

We can distinguish 3 cases, according as M, M_1 both belong to the second or to the first of the sub-sets of $L_{11} L_{12} L_{13} M M_1$, or one to the first and the other to the second of these sub-sets. In the former case, F_2 can be deformed into F_2' in such a way that the lines $a_1 a_2 a_3$ go into the lines $a_1' a_2' a_3'$, taken in this order, and $b_1 b_2 b_3$ go into any arrangement of $b_1' b_2' b_3'$; it follows that both the pairs of complementary triplets $(a_1 a_2 a_3, b_1 b_2 b_3)$, $(a_1' a_2' a_3', b_1' b_2' b_3')$ of F_2, F_2' are of the 2nd kind, and that in the previous work a_1', a_2', a_3' (as also b_1', b_2', b_3') can be considered in any order: hence the deformations of F_2 into F_2'—each of which must transform the pair $(a_1 a_2 a_3, b_1 b_2 b_3)$ into the pair $(a_1' a_2' a_3', b_1' b_2' b_3')$—are $6.6 = 36$ in number. In the second and third cases both the pairs $(a_1 a_2 a_3, b_1 b_2 b_3)$ and $(a_1' a_2' a_3', b_1' b_2' b_3')$ are of the 1st kind; but, while in the second case a_1 and a_2 are both hyperbolic, in the third case they are one elliptic and the other hyperbolic.

A particular case arises when F_2' coincides with F_2. Then we obtain the result:

The group Γ_2 inherent to a cubic surface F_2 is of order 36; *it is simply isomorphic with the direct product of* 2 *symmetric groups of degree* 3, *its transformations being uniquely defined by the property of performing an arbitrary substitution among the* 3 *elliptic left-handed lines of F_2 and an arbitrary substitution among the* 3 *elliptic right-handed lines of F_2. The group operates transitively among the* 9 *pairs of complementary triplets of the* 1*st kind of F_2, any* 2 *such pairs being transformed one into the other by* 4 *transformations of Γ_2; in particular, the sub-group of Γ_2 transforming into itself one of these pairs is trirectangular, and coincides with the sub-group of Γ_2 which leaves unchanged each of the* 2 *elliptic lines of the pair.*

43. Fig. 42 shows that the fundamental triad T inherent to the plane representation of F_2 (§ 21) is related in the following simple way to a definite one of the 9 pairs of complementary triplets of the 1st kind of F_2: one of the 3 Steiner sets of T is formed by the 9 real lines of F_2 distinct from the 6 lines of this pair. Since (§ 42) the 9 pairs of complementary triplets of the 1st kind of F_2 are equivalent with respect to Γ_2, we conclude that:

In the real domain, a cubic surface F_2 can degenerate into 3 *independent real planes in* 9 *essentially distinct ways, leading to* 9 *graphical representations for its lines, which are distinct even if confined to the* 15 *real lines.*

(iii) THE GROUP Γ_3 OF THE SURFACES F_3

44. Let us consider (in the projective complex space) 3 distinct planes α, β, γ through a line r, and 3 pairs of distinct lines $a_1 a_2$, $b_1 b_2$, $c_1 c_2$ (distinct from r) respectively belonging to α, β, γ and intersecting in A, B, C. The 7 lines $r\ a_1\ a_2\ b_1\ b_2\ c_1\ c_2$ present 19 conditions to the cubic surfaces; these conditions are independent, so that the only cubic surface through $ra_1 a_2 b_1 b_2 c_1 c_2$ is $\alpha\beta\gamma$, if (and only if) the 3 pairs of points $A_1 A_2$, $B_1 B_2$, $C_1 C_2$ intersected on r by $a_1 a_2$, $b_1 b_2$, $c_1 c_2$ do not belong to an involution. If, on the contrary, we suppose that $A_1 A_2$, $B_1 B_2$, $C_1 C_2$ are any 3 distinct pairs of points of a single (non-singular) involution $\mathfrak{J}$, then *the cubic surfaces through $ra_1 a_2 b_1 b_2 c_1 c_2$ form a pencil* Θ: in this case, in fact, one at least of the points A, B, C—say C—does not belong to r, and the lines through C incident to the 4 pairs $a_1 b_1$, $a_1 b_2$, $a_2 b_1$, $a_2 b_2$ together with c_1, c_2 are the generators of a quadric cone, so that there is a cubic surface through $ra_1 a_2 b_1 b_2 c_1 c_2$ having a double point at C (and therefore distinct from $\alpha\beta\gamma$); furthermore, we cannot have a net of cubic surfaces through $ra_1 a_2 b_1 b_2 c_1 c_2$, or else we should obtain within the net a pencil having γ as a fixed part, and therefore consisting further of a pencil of quadrics through $a_1 a_2 b_1 b_2$, which obviously cannot exist.

We suppose the three points A, B, C not to be collinear; it is then easy to prove, by means of elementary projective considerations, that the 4 quadrics containing $a_1 b_1 c_1$, $a_1 b_2 c_2$, $a_2 b_1 c_2$, $a_2 b_2 c_1$ belong to a net, having in common the line r, the points A, B, C,† and a further point P, from which the 3 pairs of lines $a_1 a_2$, $b_1 b_2$, $c_1 c_2$ project on a plane into the 3 pairs of opposite sides of a quadrangle; we obtain likewise

† We do not exclude the possibility that, for instance, A should fall on r; in this case, all the quadrics of the net acquire the fixed tangent plane α at the point A.

another point Q, by considering the 4 quadrics of a net determined by $a_2 b_2 c_2$, $a_2 b_1 c_1$, $a_1 b_2 c_1$, $a_1 b_1 c_2$. It follows that the only singular surfaces of the pencil Θ are $\alpha\beta\gamma$, the surfaces having a double point at one of the points A, B, C not belonging to r, and 2 other surfaces having a double point at P or Q. Every surface F of Θ has α, β, γ as tritangent planes: when F varies in Θ, its other 2 tritangent planes through r describe an involution, which obviously is the involution having as double planes $\pi = rP$ and $\chi = rQ$.

We denote by L, M the 2 (distinct) double points of the involution $\mathfrak{J}$, and by λ, μ the 2 planes through r which correspond to LL, MM in the projective relationship between the planes through r and the pairs of $\mathfrak{J}$ which associates $A_1 A_2$, $B_1 B_2$, $C_1 C_2$ to α, β, γ respectively. It is then easily seen that the (not necessarily distinct) points $ABCPQLM$ belong to a cubic curve touching at L, M the planes λ, μ respectively, which is the locus of the points of contact—generally not belonging to r—of the surfaces F of Θ with their variable tritangent planes through r; the 2 variable tritangent planes tend to λ and μ when F tends to $\alpha\beta\gamma$: so that λ, μ are harmonically separated by the planes π, χ.

We can determine rationally a surface F of the pencil Θ by requiring that it has a given plane δ through r as tritangent plane; F is non-singular if, and only if, δ does not coincide with one of the planes $\lambda\mu\pi\chi\alpha\beta\gamma\alpha_1\beta_1\gamma_1$, where α_1, β_1, γ_1 are the harmonic conjugates of α, β, γ with respect to π and χ.

45. We suppose now that $r\alpha\beta\gamma ABCa_1 a_2$ are real, but that the pairs of lines $b_1 b_2$, $c_1 c_2$ (and their intersections with r) are conjugate complex. The involution $\mathfrak{J}$ is necessarily hyperbolic, so that L, M, λ, μ are real; and, moreover, the points P, Q, and therefore also the planes π, χ, are real: π, χ separate λ, μ, and they can be so named that α is a plane of the angle $\lambda\pi\mu$ and β, γ are planes of the angle $\lambda\chi\mu$.†

Our figure $(r\alpha\beta\gamma a_1 a_2 b_1 b_2 c_1 c_2)$ can be continuously deformed into any other figure $(r'\alpha'\beta'\gamma' a_1' a_2' b_1' b_2' c_1' c_2')$ defined likewise, in such a way that every intermediate position is a figure of the same type. More precisely, the deformation can be performed so that $r\alpha\beta\gamma PQ$ go into $r'\alpha'\beta'\gamma' P'Q'$ respectively (P', Q' having been defined in the same way as P, Q); we have then 4 distinct kinds of deformation, in which $a_1 a_2 b_1 b_2 c_1 c_2$ are in order transformed into $a_1' a_2' b_1' b_2' c_1' c_2'$, or $a_1' a_2' b_2' b_1' c_2' c_1'$, or $a_2' a_1' b_1' b_2' c_2' c_1'$, or $a_2' a_1' b_2' b_1' c_1' c_2'$.

With our present assumptions the pencil Θ is real. Its surfaces can

† One or two (but not all) of the planes α, β, γ can coincide with one of the planes λ, μ.

be defined as at the end of § 44; and, taking also into account § 31, we see that in such a way we obtain:

(i) *A surface of the type* F_3, if δ is real, interior to the angle $\lambda\pi\mu$, and distinct from α, π, α_1.

(ii) *A surface of the type* F_4, if δ is complex and has its conjugate complex plane as harmonic conjugate with respect to π and χ.

(iii) *A surface of the type* F_5, if δ is real, interior to the angle $\lambda\chi\mu$, and distinct from β, γ, χ, β_1, γ_1.

Each of these surfaces has a fifth tritangent plane through r, which is the plane, δ_1 say, harmonic conjugate of δ with respect to π and χ.

46. If the surface F_3 considered in § 45 (i) is deformed into another surface F_3' of the same type, in such a way that no intermediate position is singular, we denote the elements of F_3' which correspond to those of F_3 by the same letters with dashes. Then r' is the only hyperbolic line of the 2nd kind of F_3', and we can suppose, without restriction, that the planes $\delta\alpha\delta_1\beta\gamma$ and $\delta'\alpha'\delta_1'\beta'\gamma'$ occur in these cyclic orders within their pencils.

Conversely, if F_3 has to be deformed into F_3', in such a way that every intermediate position is non-singular, then r must go into r', and—order apart—the planes β, γ must go into β', γ' and δ, α, δ_1 into δ', α', δ_1', since, while each of the former contains two conjugate complex lines of F_3 or F_3', each of the latter contains three real lines. The planes δ, α, δ_1 are in this order in the angle $\lambda\pi\mu$, so that α is certainly distinct from λ, μ and its two lines a_1, a_2 intersect r in a pair of distinct points A_1, A_2: this property has to be preserved during a deformation of the required type, which obviously must associate α' with α, but *a priori* can transform β, γ, δ, δ_1 either into β', γ', δ', δ_1' or into γ', β', δ_1', δ' respectively.

We shall now discuss in more detail one of these two cases, for instance the first. It is obviously possible to deform continuously the planes δ, α, δ_1, β, γ into δ', α', δ_1', β', γ' respectively; and thus we define uniquely a correspondence between the orientations of r, r'. We can suppose a_1', a_2' and their respective intersections A_1', A_2' with r' to be such that A_1LA_2M and $A_1'L'A_2'M'$ are corresponding directions of r, r'; then a_1', a_2' must respectively correspond to a_1, a_2: and we can deform a_1 into a_1' and a_2 into a_2' in such a way that the variable figure deduced from r, a_1, a_2 is always a proper triangle. Moreover, by virtue of § 45, we can extend the deformation further in two essentially distinct ways, so that $b_1b_2c_1c_2$ become either $b_1'b_2'c_1'c_2'$ or $b_2'b_1'c_2'c_1'$, and the

properties assumed for δ in (i) are preserved, which defines two distinct deformations of F_3 into F'_3 which are of the type required. We also obtain two other deformations, transforming $a_1 a_2 b_1 b_2 c_1 c_2$ either into $a'_1 a'_2 c'_1 c'_2 b'_1 b'_2$ or into $a'_1 a'_2 c'_2 c'_1 b'_2 b'_1$ respectively, corresponding to the second of the above alternatives.

In particular, if F'_3 is taken to coincide with F_3, we obtain the result:

The group Γ_3 is of order 4, and simply isomorphic with a trirectangular group; it leaves unchanged each of the 2 elliptic lines and the hyperbolic line of the 2nd kind of F_3, and performs on the other 4 real lines—which are 2 pairs of incident hyperbolic lines of the 1st kind—a transitive group having the 2 pairs of incident lines as imprimitive systems.

47. The 6 real lines of F_3 incident to its hyperbolic line of the 2nd kind can be distributed into 2 triplets in 4 distinct ways; and, by virtue of the last theorem, each of these 4 pairs of triplets is transformed into itself by 2 distinct operations of Γ_3, so that it has only one other pair of triplets as its transform by means of Γ_3.

Since the fundamental triad inherent to the plane representation of F_3, given by Fig. 43, is characterized by the property that 2 of its Steiner sets each contain a triplet of such a pair, it follows that:

In the real field, a cubic surface F_3 can degenerate into 3 independent real planes in only 2 essentially distinct ways, leading to 2 graphical representations for its lines, which are distinct even if we consider only the 7 real lines of F.

(iv) THE GROUP Γ_4 OF THE SURFACES F_4

48. A cubic surface F_4 contains 3 real lines belonging to a plane α; directing our attention to one of them, r say, we denote for the moment the other 2 by a_1, a_2 and consider the 4 further tritangent planes through r. Two of these tritangent planes, β and γ, are real and contain further a pair of conjugate complex lines of the 1st kind, $b_1 b_2$ and $c_1 c_2$; the other 2 tritangent planes, δ and δ_1, are conjugate complex and each contains further 2 complex lines of the 2nd kind. We observe that any one or two of the pairs $a_1 a_2$, $b_1 b_2$, $c_1 c_2$ may intersect on r, and that the choice of one of the 2 planes δ, δ_1 fixes the order in which we have to consider β, γ if we wish these 2 planes, together with α and the chosen plane δ or δ_1, to determine a cross-ratio having a positive imaginary part; the importance of our last remark lies (as we shall see below) in the fact that the necessary and sufficient condition that the planes $\alpha\beta\gamma\delta$ can be continuously deformed into 4 distinct planes of

a pencil, $\alpha'\beta'\gamma'\delta'$ respectively, of which $\alpha'\beta'\gamma'$ are real and δ' is complex, in such a way that there occur as intermediate positions only figures of the same kind as $\alpha\beta\gamma\delta$, is that the coefficients of i in $(\alpha\beta\gamma\delta)$ and in $(\alpha'\beta'\gamma'\delta')$ have the same sign.

We can identify our surface F_4 with the surface F_4 defined in § 45 (ii), and try to deform it into another surface F'_4 of the same type, for which we use a similar notation with dashes. If we wish r to go into r', by virtue of § 45 we have 4 distinct possibilities if, the above-mentioned condition being satisfied, we require that $\alpha\beta\gamma\delta$ go respectively into $\alpha'\beta'\gamma'\delta'$, and 4 other possibilities if we require that $\alpha\beta\gamma\delta$ go respectively into $\alpha'\gamma'\beta'\delta'_1$. Since r is any of the 3 real lines of F_4, we obtain in all 24 deformations of F_4 into F'_4.

49. If in particular F'_4 is taken to coincide with F_4, we see (taking into account §§ 45, 48) that the group Γ_4 is of order 24 and performs 6 distinct substitutions upon the 3 real lines of F_4. This group contains a self-conjugate sub-group of order 4, Δ say, given by the transformations of Γ_4 leaving each of those real lines unchanged; Δ is trirectangular and has as quotient group a symmetric group of degree 3.

In order to be able to discuss the matter in more detail, we make some preliminary remarks and slightly alter the notation.

A complex point P belonging to a self-conjugate line r determines on it a positive (projective) direction of movement, such that three distinct real points $P_1 P_2 P_3$ occur on r in this direction if, and only if, $\Im(PP_1 P_2 P_3) > 0$.† The directions induced on a self-conjugate line by two of its conjugate complex points or by two complex points corresponding in an hyperbolic involution are obviously opposite; and the necessary and sufficient condition that two complex points P, P' of a self-conjugate line r can be joined by a continuous chain belonging to r and containing no real point, is that P and P' induce on r the same direction. If we consider in [3] two incident lines, r and a say, of which the former is real and the second is complex of the 2nd kind, then the point ra induces a direction of movement on r and the plane ra induces a direction of rotation around r, so that r is completely oriented and we can accordingly say that the pair ra is *right-handed* or *left-handed*; the necessary and sufficient condition that the pair ra can be transformed into another similar pair $r'a'$, by means of a continuous deformation in [3] for which all the intermediate positions are

† We denote by $\Re(\nu)$ and $\Im(\nu)$ the real and imaginary parts respectively of the complex number ν.

similarly constituted, is that ra and $r'a'$ are both right-handed or both left-handed: this condition is satisfied if, for instance, r, r' coincide and a, a' are conjugate.

A cubic surface F_4 contains 3 real lines r_1, r_2, r_3 and 12 complex lines of the 2nd kind, forming (§ 32) a double-six of the 8th kind:

$$\omega = \begin{pmatrix} s_1 \bar{s}_1 s_2 \bar{s}_2 s_3 \bar{s}_3 \\ t_1 \bar{t}_1 t_2 \bar{t}_2 t_3 \bar{t}_3 \end{pmatrix},$$

where the 2 pairs of lines $s_i \bar{s}_i$ and $t_i \bar{t}_i$ are skew, conjugate complex, and incident to r_i ($i = 1, 2, 3$). The lines s_i, $\bar{t}_i$ are in a complex plane through r_i and intersect this line in 2 complex points, which correspond in the hyperbolic involution determined by F_4 on it (§ 27); hence $r_i s_i$, $r_i \bar{s}_i$ have concordant orientations, opposite to those of $r_i t_i$, $r_i \bar{t}_i$. Each of the 4 transformations of Γ_4 leaving r_3 unaltered and interchanging r_1, r_2, must consequently transform s_3 either into itself or into $\bar{s}_3$, so that each of them transforms the sextuplet $(s_1 \bar{s}_1 s_2 \bar{s}_2 s_3 \bar{s}_3)$ into itself. The 6 lines of this sextuplet can *a priori* be respectively transformed by these operations into one of the following 8 arrangements:

$$\begin{array}{ll} s_2 \bar{s}_2 s_1 \bar{s}_1 s_3 \bar{s}_3 & \bar{s}_2 s_2 \bar{s}_1 s_1 \bar{s}_3 s_3 \\ s_2 \bar{s}_2 \bar{s}_1 s_1 \bar{s}_3 s_3 & \bar{s}_2 s_2 s_1 \bar{s}_1 s_3 \bar{s}_3 \\ \bar{s}_2 s_2 s_1 \bar{s}_1 \bar{s}_3 s_3 & s_2 \bar{s}_2 \bar{s}_1 s_1 s_3 \bar{s}_3 \\ \bar{s}_2 s_2 \bar{s}_1 s_1 s_3 \bar{s}_3 & s_2 \bar{s}_2 s_1 \bar{s}_1 \bar{s}_3 s_3; \end{array}$$

we shall show that the 4 arrangements of the second column cannot arise, so that the 4 above-mentioned transformations of Γ_4 are consequently those determined by the arrangements of the first column.

By comparing the relations (1), (4) of § 22, we have, in fact, that

$$t_3(s_1 \bar{s}_1 s_2 \bar{s}_2) \neq \bar{t}_3(s_1 \bar{s}_1 s_2 \bar{s}_2),$$

where $t_3(s_1 \bar{s}_1 s_2 \bar{s}_2)$ and $\bar{t}_3(s_1 \bar{s}_1 s_2 \bar{s}_2)$ represent the cross-ratios of the points intersected on t_3 and $\bar{t}_3$ by s_1, $\bar{s}_1$, s_2, $\bar{s}_2$ respectively; these 2 cross-ratios being manifestly conjugate, it follows that $\mathfrak{I}[t_3(s_1 \bar{s}_1 s_2 \bar{s}_2)]$ cannot vanish. But this expression is replaced by one of opposite sign, when we perform the substitution of $\mathfrak{G}$ determined by any one of the 4 arrangements of the second column; hence such a substitution cannot belong to Γ_4.

From the above representation of the transformations of Γ_4 which interchange 2 of the lines r_1, r_2, r_3, we deduce at once the representation of the whole group Γ_4 by considering the products formed with them. Thus we see that:

The 24 *operations of the group* Γ_4 *transform into itself each of the two*

sextuplets of ω, *and are determined by the property of rearranging the* 6 *lines* $(s_1\bar{s}_1 s_2\bar{s}_2 s_3\bar{s}_3)$ *of one of them in the* 24 *following ways:*

$$\begin{gathered} s_{i_1}\bar{s}_{i_1}s_{i_2}\bar{s}_{i_2}s_{i_3}\bar{s}_{i_3} \\ s_{i_1}\bar{s}_{i_1}\bar{s}_{i_2}s_{i_2}\bar{s}_{i_3}s_{i_3} \\ \bar{s}_{i_1}s_{i_1}s_{i_2}\bar{s}_{i_2}\bar{s}_{i_3}s_{i_3} \\ \bar{s}_{i_1}s_{i_1}\bar{s}_{i_2}s_{i_2}s_{i_3}\bar{s}_{i_3}, \end{gathered}$$

where i_1, i_2, i_3 *is any permutation of the numbers* 1, 2, 3.

50. The cubic surface F_4 contains 3 self-conjugate double-sixes ω_1, ω_2, ω_3 (of the 11th kind, § 31, iv), determining together with ω a set of 4 mutually permutable σ-transformations; and we have immediately from § 49 that, if σ, σ_1, σ_2, σ_3 are the σ-transformations defined by ω, ω_1, ω_2, ω_3, then the group Δ considered at the beginning of § 49 is simply given by

$$1, \qquad \sigma_2\sigma_3, \qquad \sigma_3\sigma_1, \qquad \sigma_1\sigma_2.$$

With the notation inherent to Fig. 44 we have

$$\sigma = [333], \qquad \sigma_1 = [13], \qquad \sigma_2 = [23], \qquad \sigma_3 = [33];$$

and with that supplied by Fig. 46 we have

$$\sigma = [33], \qquad \sigma_1 = [113], \qquad \sigma_2 = [223], \qquad \sigma_3 = [333]:$$

we obtain for Γ_4 a simple geometrical representation, connected with the last figure, by noticing that Γ_4 contains a sub-group Δ' of order 6, whose transformations are represented in Fig. 46 by applying the same (arbitrary) substitution upon the indices 1, 2, 3 of 1_1, 1_2, 1_3 and 2_1, 2_2, 2_3, and that each transformation of Γ_4 can be expressed as the product of an operation of Δ by an operation of Δ'. From this geometrical representation the results which follow can be deduced almost immediately.

We recall (§ 31, iv) that the 12 complex lines of the 1st kind of F_4 form a self-conjugate 12-set (of the 11th kind), i.e. they can be intersected on F_4 by the faces of 2 conjugate complex tetrahedra, corresponding in a homology; the 4 lines intersected by the 4 pairs of corresponding faces upon the fundamental plane of the homology are real, and the planes touching F_4 in 2 opposite vertices of their quadrilateral go through one and the same real line of F_4. Then we see that:

The group Γ_4 *induces* 24 *distinct substitutions among the* 4 *sides of this quadrilateral, so that it is simply isomorphic with a symmetric group of degree* 4; *the above* 2 *tetrahedra are transformed each into itself or are interchanged by an operation of* Γ_4, *according as the substitution induced by this operation among these* 4 *sides is of even or odd class.*

51. By virtue of § 29, a cubic surface F_4 can degenerate into 3 independent planes either all real or one real and 2 conjugate complex; its graphical representation is correspondingly of the type given by Fig. 44 or by Fig. 46.

In the first case, the fundamental triad T inherent to the representation is such that each of its 3 Steiner sets contains 4 complex lines of the first kind incident to the same real line of F_4. There is only one other triad satisfying this condition (§ 32), which (Fig. 44 and § 13) is the transform T' of T by means of $\sigma = [333]$; and 12 of the 24 transformations of Γ_4 interchange T and T' (§ 49). Consequently,

In the real domain a cubic surface F_4 can degenerate into 3 *independent real planes in* 2 *essentially distinct ways, leading to* 2 *graphical representations of its lines, which are, however, distinct only as regards the complex lines.*

In the second of these 2 cases, we notice that, with the 12 complex lines of the 1st kind of F_4, we can form 8 triplets of lines no 2 of which are incident to the same real line of F_4; thus, with the notation supplied by Fig. 46, we have the following 4 pairs of complementary triplets of that type:

103, 203, 303	013, 023, 033
013, 130, 120	103, 310, 210
230, 023, 210	320, 203, 120
320, 310, 033	230, 130, 303.

Since, by virtue of § 50, Γ_4 acts transitively upon these 4 pairs (performing on them 24 distinct substitutions) and, on the other hand, the fundamental triad inherent to the plane representation of F_4 given by Fig. 46 is characterized by the property that 2 of its Steiner sets contain each a triplet of such a pair, we see that:

In the real domain a cubic surface F_4 can degenerate into 3 *independent planes, of which one π_3 is real and* 2 *are conjugate complex, in* 4 *essentially distinct ways; the* 3 *tritangent planes of the* 3*rd kind which tend to π_3 form any one of the* 4 *sets of* 3 *planes having their principal points collinear.*

(v) The group Γ_5 of the surfaces F_5

52. A cubic surface F_5 contains 3 real hyperbolic lines belonging to a plane α; directing our attention to one r of them, we fix a direction of movement around r and consider the 4 further distinct tritangent planes containing r—β, γ, δ, δ_1 say—and the parabolic planes λ, μ through r: all these planes are real and occur in the oriented pencil of

axis r, for instance, in the order $\alpha\lambda\beta\delta\delta_1\gamma\mu$. The inversion of the orientation around r is equivalent to interchanging λ and μ, β and γ, δ and δ_1, so that we see that, although the 4 tritangent planes β, δ, δ_1, γ are all of the 3rd kind, yet none of the planes β, γ can be equivalent to one of the planes δ, δ_1 with respect to the transformations of Γ_5. We call β, γ and δ, δ_1 tritangent planes *of the 1st* and *of the 2nd type* respectively;† neither of the latter can ever coincide either with λ or with μ (which is not necessarily true for the former): in other words, every such plane contains 2 conjugate complex lines of F_5, intersecting r in 2 distinct (and therefore non-real) points.

We can let F_5 degenerate into 3 independent planes, π_1, π_2, π_3, of which the first 2 are conjugate complex and the third is real (§ 29), and thus obtain the graphical representation of its lines given by Fig. 45. If, for instance, $r = 110$, namely, if—with our usual notation —when F tends to $F_0 = \pi_1\pi_2\pi_3$ the line r has $r_0 = P_{11}P_{21}$ as limit, then π_3 is the limit of α, while the planes β, δ, δ_1, γ (apart from the order) have as limits the planes π_3, r_0P_{31}, r_0P_{32}, r_0P_{33}, which occur in this or in the opposite cyclic order in their pencil. For reasons of continuity, since α, β, δ, δ_1, γ are distinct and in this order, we see that one of the two tritangent planes of the 1st type through r tends to π_3, and the other tends either to r_0P_{31} or to r_0P_{33}. Using the symbolical representation supplied by Fig. 45, we have consequently that, in any case, the tritangent planes

$$\begin{pmatrix}110\\230\\320\end{pmatrix},\qquad \begin{pmatrix}130\\220\\310\end{pmatrix},\qquad \begin{pmatrix}120\\210\\330\end{pmatrix} \qquad (*)$$

are of the 1st type, and the tritangent planes (112), (222), (332) are of the 2nd type.

53. If we identify the surface F_5 considered in § 52 with the surface F_5 defined in § 45 (iii) and apply § 45, we easily see that there are 24 essentially distinct deformations of F_5 into any other surface of the same type. In particular the group Γ_5 is of order 24, and performs 6 distinct substitutions upon the 3 real lines of F_5; the transformations of Γ_5 leaving unaltered each of these 3 lines constitute a self-conjugate trirectangular sub-group, Δ, having as quotient group a symmetric group of degree 3.

The 9 lines belonging to the planes (*) constitute a Steiner set S;

† Later on (in § 83), we shall signalize a *topological distinction* between these two types of tritangent planes of F_5.

and we shall prove that the only operation of Δ which transforms S into itself is the identity. An operation of Γ_5 transforming into themselves S and each of the real lines (110), (220), (330) must in fact transform into themselves the planes α and (*), and therefore each of the 13 real tritangent planes of F_5; the group $\mathfrak{G}$ inherent to F_5 contains only one non-identical operation satisfying this condition, which is represented by the symmetry transforming Fig. 45 into itself; but this operation interchanges the two lines 102, 012, which (§ 52) intersect $r = 110$ in two (non-real) conjugate complex points, inducing on r opposite positive directions (§ 49): therefore this operation cannot be obtained by a real circulation of F_5 (i.e. it does not belong to Γ_5), since at the same time it leaves unaltered the direction of rotation around r.

Two of the substitutions, T_1, T_2 say, of the above trirectangular group Δ interchange β, γ, namely, the two tritangent planes of the 1st type through r, which (owing to § 52) we can suppose to be (113) and the first of the planes (*). The Steiner sets S_1, S_2, transforms of S by T_1, T_2 respectively, must be distinct: since, otherwise, there would be in Δ a non-identical transformation, $T_1 T_2$, leaving S unaltered. Both S_1 and S_2 must contain the 5 lines

$$110, \qquad 220, \qquad 330, \qquad 103, \qquad 013,$$

and, by virtue of § 52, none of them can contain one of the lines

$$202, \qquad 022, \qquad 302, \qquad 032;$$

it follows that S_1 and S_2, apart from the order, must coincide with the sets belonging to the Steiner trihedra

$$(113),\ (223),\ \begin{pmatrix}120\\210\\330\end{pmatrix} \quad \text{and} \quad (113),\ \begin{pmatrix}130\\220\\310\end{pmatrix},\ (333),$$

so that the real tritangent planes (113), (223), (333) are of the 1st type, and consequently (111), (221), (331) are of the 2nd type. We can say in conclusion that:

The 24 *complex lines* (*of the* 1*st kind*) *of* F_5 *are of two different types, inequivalent with respect to the transformations of* Γ_5, *according to the type of real tritangent plane to which they belong; the lines of the* 1*st type form a self-conjugate* 12-*set* (*of the* 11*th kind*), *and those of the* 2*nd type constitute a self-conjugate double-six* (*of the* 12*th kind*).

54. Let us consider the 3 real lines of F_5, which now we call r_1, r_2, r_3, and the double-six

$$\begin{pmatrix} s_1 & t_1 & s_2 & t_2 & s_3 & t_3 \\ \bar{t}_1 & \bar{s}_1 & \bar{t}_2 & \bar{s}_2 & \bar{t}_3 & \bar{s}_3 \end{pmatrix}$$

made up of the 12 complex lines of the 2nd type of this surface, where the two pairs of lines s_i, $\bar{s}_i$ and t_i, $\bar{t}_i$ are conjugate complex of the 1st kind, and incident to r_i in two pairs of distinct complex points (§§ 52, 53), which induce two opposite positive directions on r_i. The pairs s_i, t_i and $\bar{s}_i$, $\bar{t}_i$ also define two pairs of complex points inducing opposite positive directions on r_i, since, if $F_5 \rightarrow F_0$, both these two pairs tend to the pair of conjugate complex points P_{1i}, P_{2i}, as is immediately shown by Fig. 45.

Each of the 4 transformations of Γ_5 leaving r_3 unaltered and interchanging r_1, r_2, must consequently transform s_3 either into itself or into t_3. In fact one of them cannot transform s_3 into $\bar{s}_3$, since otherwise it would leave unchanged the real tritangent plane $r_3 s_3 \bar{s}_3$, and therefore also the direction of rotation around r_3 in which this plane is followed by $r_3 t_3 \bar{t}_3$, $r_3 r_2 r_1$, so that we should have opposite complete orientations of r_3 corresponding in it; similarly, s_3 cannot be transformed into $\bar{t}_3$. Hence each of these 4 operations must transform the sextuplet $(s_1 t_1 s_2 t_2 s_3 t_3)$ into itself; and, *a priori*, each of them can only rearrange the 6 lines in one of the following 8 ways:

$$\begin{array}{ll} s_2\, t_2\, s_1\, t_1\, s_3\, t_3 & \qquad t_2\, s_2\, t_1\, s_1\, t_3\, s_3 \\ s_2\, t_2\, t_1\, s_1\, t_3\, s_3 & \qquad t_2\, s_2\, s_1\, t_1\, s_3\, t_3 \\ t_2\, s_2\, s_1\, t_1\, t_3\, s_3 & \qquad s_2\, t_2\, t_1\, s_1\, s_3\, t_3 \\ t_2\, s_2\, t_1\, s_1\, s_3\, t_3 & \qquad s_2\, t_2\, s_1\, t_1\, t_3\, s_3; \end{array}$$

we shall prove that the 4 arrangements of the second column are not in fact possible, so that the 4 above transformations are simply those determined by the arrangements of the first column.

By virtue of the relation (1) of § 22, we have, in fact, that

$$\bar{s}_3(s_1 t_1 s_2 t_2) = s_3(\bar{t}_1 \bar{s}_1 \bar{t}_2 \bar{s}_2), \qquad \bar{t}_3(s_1 t_1 s_2 t_2) = t_3(\bar{t}_1 \bar{s}_1 \bar{t}_2 \bar{s}_2),$$

so that both the numbers (finite and $\neq 0, 1$) represented by these cross-ratios are *real*; they are, moreover, *distinct*, on account of the relation (4) of § 22. If $s_1 t_1 s_2 t_2 s_3 t_3$ go into the first or last arrangement of the second column, then these 2 real numbers are interchanged, so that their difference (which is always $\neq 0$) changes its sign; and this result can certainly not be produced by means of a continuous deformation. We can dispose of the 2 remaining cases by remarking that, if we consider upon the complex line $\bar{s}_3$ the 4 distinct points cut on it by s_1, t_1, s_2, t_2, it is not possible to perform on these points an odd substitution by means of a continuous deformation, so that at each intermediate stage we have 4 distinct collinear points having a real cross-ratio.

From the representation arrived at above of the transformations of Γ_5 which interchange 2 of the lines r_1, r_2, r_3, we deduce at once the

representation of the whole group Γ_5 by considering the products formed with them. Thus we see that the *representation of* Γ_5 can be derived from that given for Γ_4 at the end of § 49, simply by substituting $t_1 t_2 t_3$ for $\bar{s}_1 \bar{s}_2 \bar{s}_3$ respectively; considerations similar to those of § 50 hold consequently for F_5, and we can say that:

Through each of the 3 *real lines of* F_5 *there are* 2 *tritangent planes of the* 3*rd kind and* 1*st type, and the principal points of these* 3 *pairs of planes are the* 3 *pairs of opposite vertices of a plane quadrilateral. The group* Γ_5 *induces* 24 *distinct substitutions among the* 4 *sides of this quadrilateral, so that it is simply isomorphic with a symmetric group of degree* 4.

55. From Fig. 45 and §§ 52, 53 we have that, when F_5 tends to $F_0 = \pi_1 \pi_2 \pi_3$, 4 real tritangent planes of F_5 tend to π_3; they are the tritangent plane $r_1 r_2 r_3$ of the 1st kind, and 3 tritangent planes of the 3rd kind and 1st type—one through each of the lines r_1, r_2, r_3—whose principal points we denote by V_{10}, V_{20}, V_{30} respectively. In the pencil of axis r_i $(i = 1, 2, 3)$ we can consider the direction of movement in which the 5 tritangent planes through r_i occur in such an order that the plane $r_1 r_2 r_3$ is followed by the plane touching F_5 at V_{i0}; then the other tritangent planes follow these 2 in a well-determined order, and we denote by V_{i1}, V_{i2}, V_{i3} respectively their principal points: the 2 first of these 3 tritangent planes are of the 2nd type, and the last is of the 1st type (§ 52).

With the notation derived in the usual manner from the representation given by Fig. 45, let us consider the 3 real points

$$P_1 = P_{12} P_{23} . P_{13} P_{22}, \qquad P_2 = P_{13} P_{21} . P_{11} P_{23}, \qquad P_3 = P_{11} P_{22} . P_{12} P_{21},$$

which are collinear by Pappus's theorem; then § 53 and Fig. 45 show that

$$\lim_{F_5 \to F_0} V_{i0} = P_i, \qquad \lim_{F_5 \to F_0} V_{ij} = P_{3j} \quad (i, j = 1, 2, 3),$$

and that *the* 16 *lines joining* 3 *by* 3 *the vertices of the desmic tetrahedra* $V_{10} V_{11} V_{12} V_{13}$, $V_{20} V_{21} V_{22} V_{23}$, $V_{30} V_{31} V_{32} V_{33}$ (§ 31, v) *are:*

$$\begin{array}{cccccc}
 & V_{10} V_{21} V_{31}, & V_{11} V_{20} V_{31}, & V_{11} V_{21} V_{30}, & V_{12} V_{23} V_{31}, & V_{11} V_{23} V_{32}, \\
V_{10} V_{20} V_{30}, & V_{10} V_{22} V_{32}, & V_{12} V_{20} V_{32}, & V_{12} V_{22} V_{30}, & V_{13} V_{21} V_{32}, & V_{12} V_{21} V_{33}, \\
 & V_{10} V_{23} V_{33}, & V_{13} V_{20} V_{33}, & V_{13} V_{23} V_{30}, & V_{11} V_{22} V_{33}, & V_{13} V_{22} V_{31}.
\end{array}$$

Using the same argument as that developed at the end of § 51, we see, moreover, that:

In the real domain a cubic surface F_5 *can degenerate into* 3 *independent planes (of which one,* π_3*, is real and two are conjugate complex) in* 4 *essentially distinct ways; correspondingly, the* 3 *tritangent planes of the*

3rd kind which tend to π_3 form any one of the 4 sets of 3 planes of the 1st type having their principal points collinear.

X. The real sextuplets of lines and the primary sets of 6 coplanar points

56. We consider now the problem of *classifying the real double-sixes* (namely, the double-sixes consisting of 12 real lines), with respect to the continuous deformations of the real non-singular cubic surface defined by them; we propose at the same time to determine their *groups*, and to characterize the various types of double-sixes in connexion with the different possibilities that the general *construction* given in § 21 may present in the real domain. A partial solution of this problem has already been given, in § 38, for the double-sixes of the 1st kind.

In this and the next paragraph we shall establish in advance some results which will prove useful for our purpose.

Let us consider in [3] the figure $\mathfrak{F}$ formed by 5 distinct real points $A_1 A_2 A_3 A_4 A_5$ belonging to a line b_6, with which we associate respectively 5 distinct real planes $\alpha_1 \alpha_2 \alpha_3 \alpha_4 \alpha_5$ through this same line. We can completely orientate the line b_6, and suppose that the planes $\alpha_1 \alpha_2 \alpha_3 \alpha_4 \alpha_5$ occur in their pencil in this cyclic order. If the cyclic order of the 5 points on b_6 is $A_{i_1} A_{i_2} A_{i_3} A_{i_4} A_{i_5}$, we start from a regular convex pentagon whose vertices we denote in succession by 1, 2, 3, 4, 5, and consider the *representative pentagon of* $\mathfrak{F}$, namely, the pentagon which has i_1, i_2, i_3, i_4, i_5 as successive vertices; the form of this pentagon can only be one of those given by Figs. 51–4, and we can always reduce to one of these 4 cases by performing on the indices 1, 2, 3, 4, 5 a convenient cyclic substitution; then we say that $\mathfrak{F}$ is of *class I, II, III*, or *IV* respectively. It can be easily proved that

The necessary and sufficient condition that a figure $\mathfrak{F}$ can be continuously deformed into another similar figure $\mathfrak{F}'$ or into the transform of this by means of a negative homography, so that each intermediate figure consists of 5 distinct points of a line and 5 distinct planes through the same line related in a one-to-one correspondence, is that $\mathfrak{F}$ and $\mathfrak{F}'$ belong to the same class.

With our previous assumption for $\mathfrak{F}$, the 5 cross-ratios

$$(\alpha_1) = (\alpha_2 \alpha_3 \alpha_4 \alpha_5), \qquad (\alpha_2) = (\alpha_3 \alpha_4 \alpha_5 \alpha_1), \qquad (\alpha_3) = (\alpha_4 \alpha_5 \alpha_1 \alpha_2),$$
$$(\alpha_4) = (\alpha_5 \alpha_1 \alpha_2 \alpha_3), \qquad (\alpha_5) = (\alpha_1 \alpha_2 \alpha_3 \alpha_4)$$

are always all (finite and) > 1. We denote by (A_1), (A_2), (A_3), (A_4), (A_5)

respectively the 5 cross-ratios formed by the corresponding points A, so that, for instance, $(A_1) = (A_2 A_3 A_4 A_5)$, and remark that, according

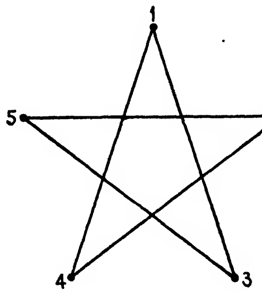


FIG. 51

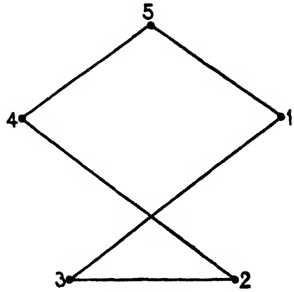


FIG. 52

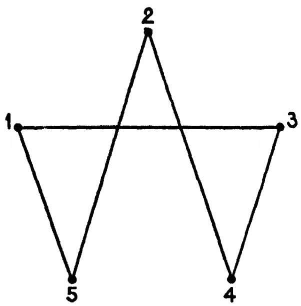


FIG. 53

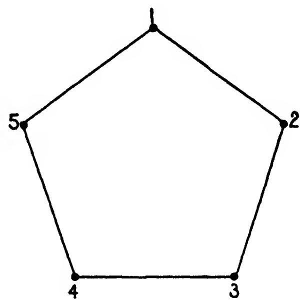


FIG. 54

as the representative polygon of $\mathfrak{F}$ is given by Fig. 51, or 52, or 53, or 54, we have respectively:

I: $(A_1) < 0,\quad (A_2) < 0,\quad (A_3) < 0,\quad (A_4) < 0,\quad (A_5) < 0,$

II: $0 < (A_1) < 1,\quad 1 < (A_2),\quad 1 < (A_3),\quad 0 < (A_4) < 1,\quad (A_5) < 0,$

III: $(A_1) < 0,\quad 1 < (A_2),\quad (A_3) < 0,\quad 0 < (A_4) < 1,\quad 0 < (A_5) < 1,$

IV: $1 < (A_1),\quad 1 < (A_2),\quad 1 < (A_3),\quad 1 < (A_4),\quad 1 < (A_5).$

In connexion with the general construction given for a double-six in § 21, we now wish to obtain all the cases a figure $\mathfrak{F}$ may offer, if we impose upon it the further conditions

$$(A_1) \neq (\alpha_1),\quad (A_2) \neq (\alpha_2),\quad (A_3) \neq (\alpha_3),\quad (A_4) \neq (\alpha_4),\quad (A_5) \neq (\alpha_5). \quad (*)$$

All these conditions are automatically satisfied if $\mathfrak{F}$ is of class I; and we obviously have only 2 possibilities if $\mathfrak{F}$ is of class III, according as either of the following 2 inequalities holds:

$$\mathrm{III}_1\colon\ (A_2) < (\alpha_2), \qquad \mathrm{III}_2\colon\ (A_2) > (\alpha_2).$$

As for the remaining cases, we observe that three consecutive (A)'s or (α)'s are connected by the following identities:

$$[(A_1)(A_3)-1][(A_2)-1] = [(\alpha_1)(\alpha_3)-1][(\alpha_2)-1] = 1;$$

hence in case II, since $(\alpha_1) > 1 > (A_1)$, we cannot have at the same time $(\alpha_2) > (A_2) > 1$, $(\alpha_3) > (A_3) > 1$; and in case IV we cannot have at the same time either $(\alpha_1) > (A_1)$, $(\alpha_2) > (A_2)$, $(\alpha_3) > (A_3)$, or $(\alpha_1) < (A_1)$, $(\alpha_2) < (A_2)$, $(\alpha_3) < (A_3)$. Therefore, in the former case the differences $(\alpha_2)-(A_2)$ and $(\alpha_3)-(A_3)$ can only be either both negative or one negative and the other positive, so that we obtain essentially only 2 possibilities, given by

$$\mathrm{II}_1\colon\ (A_2) > (\alpha_2), \qquad (A_3) > (\alpha_3),$$
$$\mathrm{II}_2\colon\ (A_2) > (\alpha_2), \qquad (A_3) < (\alpha_3).$$

In the latter case three consecutive differences $(\alpha_i)-(A_i)$ ($i = 1, 2,\ldots, 5$, the successor of $i = 5$ being $i = 1$) can never have the same sign: so that, consequently, two of them which are non-consecutive have one sign and the other 3 have the opposite sign, and we obtain essentially only two possibilities, given by

$$\mathrm{IV}_1\colon\ (A_1) > (\alpha_1),\ \ (A_2) > (\alpha_2),\ \ (A_3) < (\alpha_3),\ \ (A_4) > (\alpha_4),\ \ (A_5) < (\alpha_5),$$
$$\mathrm{IV}_2\colon\ (A_1) < (\alpha_1),\ \ (A_2) < (\alpha_2),\ \ (A_3) > (\alpha_3),\ \ (A_4) < (\alpha_4),\ \ (A_5) > (\alpha_5).$$

57. Let us consider, in the plane of an irreducible conic $\mathfrak{C}$, 4 lines n_1, n_2, n_3, n_4 of a pencil, whose centre N does not belong to $\mathfrak{C}$; and suppose that n_1, n_2 intersect $\mathfrak{C}$ in N_1 and N_1', N_2 and N_2' respectively, and that N_3, N_4 are among the intersections of $\mathfrak{C}$ with n_3, n_4 respectively. We shall prove the following identity:

$$(N_1 N_2 N_3 N_4).(N_1' N_2' N_3 N_4) = (n_1 n_2 n_3 n_4),$$

where the cross-ratios of the left-hand side are to be determined on $\mathfrak{C}$.†

To see this, we represent $\mathfrak{C}$ parametrically by means of the equations

$$x_0 : x_1 : x_2 = 1 : t : t^2,$$

† If in particular n_1, n_2 are two tangents of $\mathfrak{C}$, so that $N_1' = N_1$ and $N_2' = N_2$, this identity becomes $(N_1 N_2 N_3 N_4)^2 = (n_1 n_2 n_3 n_4)$, and expresses a well-known projective extension of the relation between an angle inscribed in a circle and the angle at the centre.

and suppose the points N_1, N_1', N_2, N_2', N_3, N_4 to correspond respectively to the values t_1, t_1', t_2, t_2', t_3, t_4 of the parameter. Hence the equations of n_1, n_2 are

$$f_1(x) \equiv t_1 t_1' x_0 - (t_1+t_1')x_1 + x_2 = 0,$$
$$f_2(x) \equiv t_2 t_2' x_0 - (t_2+t_2')x_1 + x_2 = 0;$$

we have, moreover,

$$(n_1 n_2 n_3 n_4) = \frac{f_2(z)}{f_1(z)} : \frac{f_2(y)}{f_1(y)},$$

where $(y) = (1, t_3, t_3^2)$, $(z) = (1, t_4, t_4^2)$ are the coordinates of N_3, N_4: whence our identity follows immediately, since

$$f_1(y) = (t_3-t_1)(t_3-t_1'), \qquad f_2(y) = (t_3-t_2)(t_3-t_2'),$$
$$f_1(z) = (t_4-t_1)(t_4-t_1'), \qquad f_2(z) = (t_4-t_2)(t_4-t_2').$$

As a corollary of the above identity, we have that:

If all the elements considered above are real, and, moreover, on $\mathfrak{C}$ the pairs $N_1 N_2$, $N_3 N_4$ do not separate one another, the necessary and sufficient condition that $(n_1 n_2 n_3 n_4) > (N_1 N_2 N_3 N_4)$ is that the points N_1', N_2', N_3, N_4 occur on $\mathfrak{C}$ in this cyclic order.

We obtain a $(1, 2)$-correspondence between the pencil of centre N and the conic $\mathfrak{C}$, by associating with any point of $\mathfrak{C}$ the line of the pencil through it; and it is easily seen that all the results of this paragraph can be enunciated as general properties of $(1, 2)$-correspondences between two rational curves. The first of these can be extended further as follows.

If we have a $(1, k)$-correspondence between two rational curves $\mathfrak{c}$, $\mathfrak{C}$, transforming two points n_1 and n_2 of $\mathfrak{c}$ in two sets of k points $N_1^{(1)} N_1^{(2)} \ldots N_1^{(k)}$ and $N_2^{(1)} N_2^{(2)} \ldots N_2^{(k)}$ of $\mathfrak{C}$, and two points N_3, N_4 of $\mathfrak{C}$ in two points n_3, n_4 of $\mathfrak{c}$, then

$$\prod_{i=1}^{k} (N_1^{(i)} N_2^{(i)} N_3 N_4) = (n_1 n_2 n_3 n_4),$$

where the cross-ratios of the left- and right-hand sides of the equation have to be calculated on $\mathfrak{C}$ and $\mathfrak{c}$ respectively.

58. Let us represent the lines of a cubic surface F_1 with the Schläfli notation, as in § 22; on F_1 we have 15 *double-sixes of the 2nd kind* (§ 31, i), one of which is, for instance,

$$\delta_{56} = \begin{pmatrix} c_{51} & c_{52} & c_{53} & c_{54} & a_6 & b_6 \\ c_{61} & c_{62} & c_{63} & c_{64} & a_5 & b_5 \end{pmatrix}.$$

If we likewise consider another double-six of the 2nd kind

$$\delta_{56}' = \begin{pmatrix} c_{51}' & c_{52}' & c_{53}' & c_{54}' & a_6' & b_6' \\ c_{61}' & c_{62}' & c_{63}' & c_{64}' & a_5' & b_5' \end{pmatrix},$$

belonging to a cubic surface F_1', by virtue of § 38 we have 4 essentially distinct continuous deformations of F_1 into F_1', changing the lines $a_5 a_6 b_5 b_6$ (apart from the order) into $a_5' a_6' b_5' b_6'$, and therefore transforming δ_{56} into δ_{56}'. In particular, if F_1' coincides with F_1, we see that the 60 operations of the group Γ_1 act *transitively* upon the 15 double-sixes of the 2nd kind of F_1, each of which is transformed into itself by a group Δ of 4 such operations.

In order to determine precisely this group Δ, we remark that—as a consequence of § 38—there is one (and only one) involutory operation of Γ_1 which transforms into itself a given elliptic line r of F_1 and interchanges 2 distinct lines arbitrarily chosen among the 5 elliptic lines incident to r (so that one of the remaining 3 lines is fixed and the other 2 are interchanged). If the involutory transformation of Γ_1 which leaves a_1 unaltered and interchanges b_5, b_6 has, for instance, b_4 as fixed line incident to a_1, then we consider the non-identical involutory operation of Γ_1 which leaves a_2, a_3 unaltered; this can only be transformed into itself by the previous transformation: therefore it must interchange b_5, b_6 (as well as b_1, b_4), i.e. it belongs to Δ, which consequently is *trirectangular*. The three non-identical operations of Δ act on the 12 lines of δ_{56} by interchanging them respectively with

$$\begin{pmatrix} c_{61} & c_{63} & c_{62} & c_{64} & a_5 & b_5 \\ c_{51} & c_{53} & c_{52} & c_{54} & a_6 & b_6 \end{pmatrix}, \qquad \begin{pmatrix} c_{64} & c_{62} & c_{63} & c_{61} & a_5 & b_5 \\ c_{54} & c_{52} & c_{53} & c_{51} & a_6 & b_6 \end{pmatrix}, \qquad \begin{pmatrix} c_{54} & c_{53} & c_{52} & c_{51} & a_6 & b_6 \\ c_{64} & c_{63} & c_{62} & c_{61} & a_5 & b_5 \end{pmatrix};$$

and we see that only the last such operation transforms each sextuplet of δ_{56} into itself. It is, moreover, clear that *the trirectangular group Δ is merely the sub-group consisting of the operations of Γ_1 which leave fixed the hyperbolic line c_{56}*.

With the notation of § 38 we have that the lines $c_{16} c_{26} c_{36} c_{46} a_5$ determine together with b_6 the planes $\alpha_1 \alpha_2 \alpha_3 \alpha_4 \alpha_5$ and the points $A_1' A_2' A_3' A_4' A_5$ respectively, and that, while the former occur in their pencil in the cyclic order $\alpha_1 \alpha_2 \alpha_3 \alpha_4 \alpha_5$, the latter occur on r in the cyclic order $A_1' A_3' A_2' A_4' A_5$ (§ 35). We have, moreover, a $(1, 2)$-correspondence between the planes through b_6 and their points of contact with F_1 on b_6, associating A_i, A_i' to α_i ($i = 1, 2, 3, 4, 5$); since on b_6 the pairs $A_3' A_4'$, $A_5 A_1'$ and the pairs $A_4' A_5$, $A_1' A_2'$ do not separate, and, moreover, the points $A_3 A_4 A_5 A_1'$ and $A_4 A_5' A_1' A_2'$ do not occur in these cyclic arrangements, by virtue of § 57 both the inequalities

$$(A_3' A_4' A_5 A_1') > (\alpha_3 \alpha_4 \alpha_5 \alpha_1), \qquad (A_4' A_5 A_1' A_2') > (\alpha_4 \alpha_5 \alpha_1 \alpha_2)$$

hold. Hence *the 5 planes and 5 points determined by an elliptic line of*

a double-six of the 2nd kind with the 5 *lines of the double-six incident to it, always present the case* II_1 (§ 56).

In a similar manner, taking also into account § 36, we see that *the* 5 *planes and* 5 *points determined by a hyperbolic line of a double-six of the 2nd kind together with the* 5 *lines of the double-six incident to it, always present the case* II_2.

59. Let us consider a triplet of elliptic lines of F_1, for instance $a_1 a_2 a_3$. By virtue of §§ 38, 58, the only transformation of Γ_1 having the three lines a_1, a_2, a_3 as fixed is the identity; and Γ_1 contains 3 involutory transformations, which we denote by T_1, T_2, T_3, having respectively as fixed lines the 1st, the 2nd, or the 3rd of these lines and interchanging the other two. Each of the transformations must transform into itself the triplet $b_4 b_5 b_6$ complementary to $a_1 a_2 a_3$, so that it leaves unchanged one of the lines b_4, b_5, b_6 and permutes the other two; we can therefore suppose, for instance, that T_1, T_2, T_3 have b_4, b_5, b_6 respectively as fixed lines, and say that the two complementary triplets are intrinsically related by a one-to-one correspondence associating b_4 to a_1, b_5 to a_2, b_6 to a_3. It follows that Γ_1 contains 6 distinct operations transforming the triplet $a_1 a_2 a_3$ into itself, given by T_1, T_2, T_3 and their products, which form a group *simply isomorphic with a symmetric group of degree* 3; there is one such operation rearranging $a_1 a_2 a_3$ in any given order $a_{i_1} a_{i_2} a_{i_3}$, and it correspondingly transforms $b_1 b_2 b_3$, $b_4 b_5 b_6$, $a_4 a_5 a_6$ into $b_{i_1} b_{i_2} b_{i_3}$, $b_{3+i_1} b_{3+i_2} b_{3+i_3}$, $a_{3+i_1} a_{3+i_2} a_{3+i_3}$ respectively.

If we consider any triplet $b_4 b_5 b_6$ of elliptic lines of F_1 and one, b_6, of its lines, and if we denote by A_i and α_i the point and plane determined by b_6 and a_i $(i = 1, 2, \ldots, 5)$, two cases may arise according as the points $A_1 A_2 A_3$ are either all in one, or two in one and the remaining in the other, of the two segments determined on b_6 by A_4 and A_5. By virtue of § 38, in the former case $\alpha_1 \alpha_2 \alpha_3$ are such that two lie in one and the third in the other of the two angles $\alpha_4 \alpha_5$; and in the latter case $\alpha_1 \alpha_2 \alpha_3$ are all in the same angle $\alpha_4 \alpha_5$; moreover, owing to the properties established above, we always obtain the same behaviour if we permute arbitrarily the 3 lines $b_4 b_5 b_6$, so that this behaviour only depends upon the triplet $b_4 b_5 b_6$. We call this triplet *of the* 1*st* or *of the* 2*nd type*, according as the 1st or the 2nd of the two cases considered arises. It is immediately seen that three triplets such as $b_{i_2} b_{i_3} b_6$, $b_{i_3} b_{i_1} b_6$, $b_{i_1} b_{i_2} b_6$ can never be all of the same type.

There is no restriction in supposing the planes $\alpha_1 \alpha_2 \alpha_3 \alpha_4 \alpha_5$ to occur around b_6 in this cyclic order; then, by virtue of §§ 22 and 35, the points

$B_i = b_i a_6$ occur on a_6 in the cyclic order $B_1 B_2 B_3 B_4 B_5$, so that, according as $b_{i_1} b_{i_2} b_6$ is of the 1st or 2nd type, $a_{i_1} a_{i_2} a_6$ is of the 2nd or 1st type respectively. With our assumption we have therefore that the triplets

$$\begin{array}{lllll} b_1 b_2 b_6, & b_2 b_3 b_6, & b_3 b_4 b_6, & b_4 b_5 b_6, & b_5 b_1 b_6, \\ a_1 a_3 a_6, & a_3 a_5 a_6, & a_5 a_2 a_6, & a_2 a_4 a_6, & a_4 a_1 a_6 \end{array}$$

are of the 1st type, and the triplets

$$\begin{array}{lllll} b_1 b_3 b_6, & b_3 b_5 b_6, & b_5 b_2 b_6, & b_2 b_4 b_6, & b_4 b_1 b_6, \\ a_1 a_2 a_6, & a_2 a_3 a_6, & a_3 a_4 a_6, & a_4 a_5 a_6, & a_5 a_1 a_6 \end{array}$$

are of the 2nd type. By virtue of a previous remark we see, moreover, that, since both $b_1 b_3 b_6$ and $b_3 b_5 b_6$ are of the 2nd type, the triplet $b_1 b_3 b_5$ can only be of the 1st type; more generally, it follows that the triplets

$$\begin{array}{lllll} b_2 b_4 b_5, & b_4 b_1 b_2, & b_1 b_3 b_4, & b_3 b_5 b_1, & b_5 b_2 b_3, \\ a_3 a_4 a_5, & a_4 a_5 a_1, & a_5 a_1 a_2, & a_1 a_2 a_3, & a_2 a_3 a_4 \end{array}$$

are of the 1st type, and the triplets

$$\begin{array}{lllll} b_3 b_4 b_5, & b_4 b_5 b_1, & b_5 b_1 b_2, & b_1 b_2 b_3, & b_2 b_3 b_4, \\ a_2 a_4 a_5, & a_4 a_1 a_2, & a_1 a_3 a_4, & a_3 a_5 a_1, & a_5 a_2 a_3 \end{array}$$

are of the 2nd type. Hence we deduce that in every case two complementary triplets are of the same type, and two residual triplets—namely, two triplets which together constitute a sextuplet—are of different types. Taking also into account § 38, we can say in conclusion that:

The 12 *elliptic lines of a cubic surface F_1 form a double-six* δ, *and can be distributed into* 20 *pairs of complementary triplets; in each pair, one triplet is formed by left-handed lines and the other by right-handed lines, the* 2 *triplets being intrinsically related in one-to-one correspondence. Ten of the* 20 *pairs of complementary triplets are of the* 1*st type* (*i.e. each of them consists of two triplets of the* 1*st type*), *and the other* 10 *are of the* 2*nd type. While two pairs of complementary triplets of different type are inequivalent with respect to* Γ_1, *this group acts transitively upon each set of* 10 *pairs of the same type; more precisely, there are* 6 *operations of* Γ_1 *transforming a given pair of complementary triplets into another pair of the same type: each of these operations transforms the one-to-one correspondence connecting the two triplets of the first pair, into the one-to-one correspondence connecting the two triplets of the second pair, and is characterized by the property of inducing between the left-handed* (*or between the right-handed*) *lines of the two pairs a one-to-one correspondence, which may be arbitrarily chosen. Two pairs of complementary triplets*

which are residual with respect to δ are always of different types, and are transformed each into itself by the same sub-group of Γ_1 of order 6, simply isomorphic with a symmetric group of degree 3.

60. A double-six of the 3rd kind belongs to a cubic surface F_1, and contains 6 elliptic lines constituting 2 complementary triplets (§ 31, i); according to the type of this pair of triplets (§ 59), we say that the double-six is *of the* 1*st* or *of the* 2*nd type*. On F_1 there are 10 *double-sixes of the* 3*rd kind and* 1*st type*, and 10 *double-sixes of the* 3*rd kind and* 2*nd type*; the former and the latter are strictly related with the 10 non-principal tritangent planes of the 1st kind, and with the 10 pairs of principal planes of F_1, as we shall see later on in § 65 and in § 76. From § 59 follows, moreover, at once that:

Two double-sixes of the 3*rd kind can be continuously deformed one into the other if, and only if, they are of the same type; in this case, the existing deformations operate upon the lines of the* 2 *double-sixes in* 6 *distinct ways, directly suggested by* § 59.

With the notation of §§ 35, 59 we have, for instance, that

$$\begin{pmatrix} a_1\, a_2\, a_3\, c_{56}\, c_{64}\, c_{45} \\ c_{23}\, c_{31}\, c_{12}\, b_4\, b_5\, b_6 \end{pmatrix}$$

is a double-six of *the* 3*rd kind and* 1*st type*, and that the lines $a_1\, a_2\, a_3\, c_{64}\, c_{56}$ determine together with the elliptic line b_6 the planes $\alpha_1\, \alpha_2\, \alpha_3\, \alpha_4\, \alpha_5$ and the points $A_1\, A_2\, A_3\, A'_4\, A'_5$ respectively. By virtue of §§ 35, 57, these points occur on b_6 in the cyclic order $A_1\, A_3\, A'_4\, A_2\, A'_5$ and also

$$(\alpha_3\, \alpha_4\, \alpha_5\, \alpha_1) > (A_3\, A'_4\, A'_5\, A_1),$$

so that *these* 5 *planes and* 5 *points present the case* III_1 (§ 56). Likewise we see that the double-six

$$\begin{pmatrix} a_1\, a_2\, a_4\, c_{56}\, c_{63}\, c_{35} \\ c_{24}\, c_{41}\, c_{12}\, b_3\, b_5\, b_6 \end{pmatrix}$$

is of *the* 3*rd kind and* 2*nd type*; the lines $a_1\, a_2\, c_{63}\, a_4\, c_{56}$ determine together with the elliptic line b_6 the planes $\alpha_1\, \alpha_2\, \alpha_3\, \alpha_4\, \alpha_5$ and the points $A_1\, A_2\, A'_3\, A_4\, A'_5$ respectively, which occur in these circular orders: and further

$$(\alpha_2\, \alpha_3\, \alpha_4\, \alpha_5) > (A_2\, A'_3\, A_4\, A'_5), \qquad (\alpha_3\, \alpha_4\, \alpha_5\, \alpha_1) > (A'_3\, A_4\, A'_5\, A_1),$$
$$(\alpha_4\, \alpha_5\, \alpha_1\, \alpha_2) < (A_4\, A'_5\, A_1\, A_2), \qquad (\alpha_5\, \alpha_1\, \alpha_2\, \alpha_3) > (A'_5\, A_1\, A_2\, A'_3),$$
$$(\alpha_1\, \alpha_2\, \alpha_3\, \alpha_4) < (A_1\, A_2\, A'_3\, A_4),$$

so that *these* 5 *planes and* 5 *points present the case* IV_1.

In a similar manner, taking also into account § 36, we see that *the*

5 planes and 5 points determined by an hyperbolic line of a double-six of the 3rd kind together with the 5 lines of the double-six incident to it, present either the case III_2 *or the case* IV_2 *according as the double-six is of the 1st or 2nd type.*

61. From §§ 38, 58, 60 follows immediately that:

The real double-sixes of [3] *constitute 4 distinct mutually exclusive continuous systems, given by the double-sixes of the 1st kind, those of the 2nd kind, those of the 3rd kind and 1st type, and those of the 3rd kind and 2nd type. Corresponding to all the circulations of a double-six within one of these 4 continuous systems we obtain a group of substitutions among its lines, determined above in the 4 different cases and of order* 60, 4, 6, 6 *respectively. The double-six defined by 5 lines* $a_1 a_2 a_3 a_4 a_5$ *incident to a single line* b_6, *and satisfying the conditions* (i), (ii) *of* § 21, *belong to the 1st, 2nd, 3rd, or 4th continuous system in accordance with the class* (I, II, III, *or* IV *respectively) of the figure* $\mathfrak{F}$ *formed by the 5 points* $a_i b_6$ *and the 5 planes* $a_i b_6$ $(i = 1, 2, 3, 4, 5)$. *In the first case* b_6 *is necessarily elliptic; in each of the other 3 cases* b_6 *is elliptic or hyperbolic according as* $\mathfrak{F}$ *presents the 1st or the 2nd of the two alternatives considered in* § 56.

Whence it can be deduced without difficulty that:

The transform by means of an arbitrary projectivity of a real double-six is still a real double-six of the same kind and type. Any 2 corresponding lines of the 2 double-sixes are both elliptic or both hyperbolic, if the projectivity is a homography; this is also true if the projectivity is a reciprocity and the double-sixes are of the 1st or 2nd kind; while if the projectivity is a reciprocity and the double-sixes are of the 3rd kind, any 2 corresponding lines are one elliptic and the other hyperbolic.

62. From the previous results we can deduce several interesting properties concerning what we call the *primary sets of 6 coplanar points*, namely, the sets of 6 distinct real points of a projective plane not belonging to the same conic, and no 3 of which are collinear.

If we have such a set $A_1 A_2 A_3 A_4 A_5 A_6$, then the projective image of the web of cubic curves through it is a non-singular cubic surface, F_1, defined by the set, apart from a projective transformation; on F_1, corresponding to the 6 points A_i, we obtain 6 lines a_i, constituting a sextuplet. Conversely, if a sextuplet of 6 real lines $a_1 a_2 a_3 a_4 a_5 a_6$ is given, we consider, on the cubic surface F_1 defined by it, the 15 cubic curves consisting of any two distinct lines a_i and the line of F_1 incident with them but not incident with the other 4 lines of the sextuplet;

these 15 curves belong to a homaloidal net, whose projective image gives a representation of F_1 upon a plane, transforming the 6 lines a_i into 6 points A_i: thus we obtain a primary set $A_1 A_2 A_3 A_4 A_5 A_6$, defined by the sextuplet apart from a projective transformation.†

By means of this simple relation between the sextuplets of lines and the primary sets of points, we derive from § 61 the following propositions. If we consider a primary set of points $A_1 A_2 A_3 A_4 A_5 A_6$ and one of its points, for instance A_6, we have a one-to-one correspondence between the 5 points A_i $(i = 1, 2, \ldots, 5)$ and the 5 lines $A_i A_6$; the former occur in a cyclic order upon the conic to which they belong, and the latter also occur in a certain cyclic order in the pencil of centre A_6: we can therefore—by proceeding as in § 56—define the *class* of such a correspondence [which is I, II, III, or IV according as its representative pentagon is given by Fig. 51, or 52, or 53, or 54 respectively], and say that this is the *class of* A_6 within the primary set. Then we have that:

All the 6 *points of a primary set have within it the same class, which we can call the class of the set. The primary sets of a plane constitute* 4 *distinct continuous systems*, I, II, III, IV, *whose sets have all the same class*, I, II, III, IV *respectively: in other words, the necessary and sufficient condition that* 2 *primary sets can be continuously deformed one into the other, in such a way that each intermediate position is still a primary set, is that the* 2 *sets have the same class.*

Figs. 55, 56, 57, 58, where the polygons drawn with a continuous stroke are regular, represent simple specimens of primary sets $A_1 A_2 A_3$ $A_4 A_5 A_6$, which are of class I, II, III, IV respectively. These figures clearly illustrate the following general facts, which also follow at once from § 61.

A primary set of class I *is such that each of its* 6 *points is interior to the conic through the other* 5; *its* 6 *points undergo the substitutions of an icosahedral group, if the set accomplishes all the circulations within the system* I (the rotations and symmetries transforming the set represented by Fig. 55 into itself give the sub-group of order 10 of the above-mentioned group for which A_6 is fixed).

A primary set of class II *has* 2 (*and only* 2) *points, each of which is interior to the conic through the other* 5; *each of these* 2 *points returns to*

† For the study of the above plane representation, cf., for instance, H. F. Baker, *Principles of Geometry*, vol. iii (Cambridge Univ. Press, 1934), p. 189; F. Enriques and O. Chisini, *Lezioni sulla teoria geometrica delle equazioni e delle funzioni algebriche*, vol. iii (Bologna, Zanichelli, 1924), §§ 51, 52.

itself if the set accomplishes all the circulations within the system II, *while the other* 4 *points either return each to itself or are interchanged in pairs in a well-defined manner* (A_1 and A_2 are the former points inherent to the set represented by Fig. 56, and the only non-identical trans-

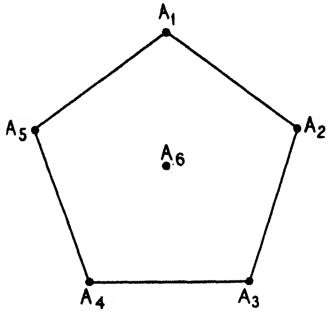


FIG. 55

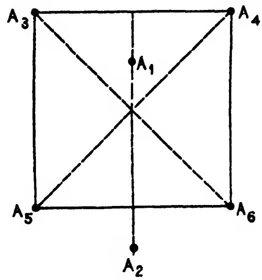


FIG. 56

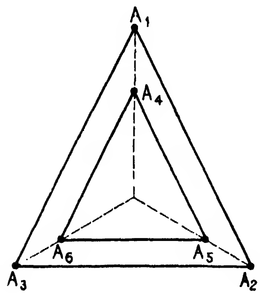


FIG. 57

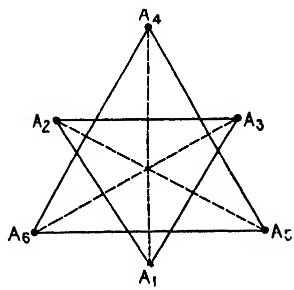


FIG. 58

formation of this set into itself is given by the symmetry with respect to the line $A_1 A_2$).

A primary set of class III *or of class* IV *has* 3 (*and only* 3) *points each of which is interior to the conic through the other* 5*; if the set is subjected to all the circulations within the system* III *or* IV *respectively, then its* 6 *points are submitted to the substitutions of a group of order* 6, *permuting those* 3 *points in* 6 *distinct ways* (the points in question being A_1, A_2, and A_3 both in Fig. 57 and in Fig. 58; the groups inherent to the sets represented by these figures are simply given by the rotations and symmetries transforming the sets into themselves).

XI. The topology of the real cubic surfaces in connexion with their real lines

63. Let us consider in a plane a pencil Σ of real curves, one $\mathfrak{C}$ of which has a double point at a non-base point P of Σ; then the neighbourhood of $\mathfrak{C}$ in Σ is decomposed by $\mathfrak{C}$ into two parts, whose curves have distinct well-defined behaviours in the neighbourhood of P. More precisely, if $\mathfrak{C}$ has a node at P, the neighbourhood $\mathfrak{N}$ of P in the plane is divided by $\mathfrak{C}$ into two pairs of opposite angular regions; each curve of Σ (distinct from but) sufficiently near $\mathfrak{C}$ has in $\mathfrak{N}$ two arcs belonging to two opposite angular regions, such that the arcs of two curves of Σ belong to distinct regions or to the same regions, according as the two curves (in the neighbourhood of $\mathfrak{C}$ in Σ) are or are not separated by $\mathfrak{C}$. If, on the contrary, $\mathfrak{C}$ has in P an isolated double point, then the curves of Σ near $\mathfrak{C}$ either have no real point in $\mathfrak{N}$, or have in $\mathfrak{N}$ a small oval containing P as an interior point; thus, if $f = 0$, $g = 0$ are the equations of $\mathfrak{C}$ and of another curve of Σ, and if, for instance, both f and g are positive in $\mathfrak{N}-P$, then the curves $f+\lambda g = 0$ of Σ having $|\lambda|$ sufficiently small belong to the first category if $\lambda > 0$ and to the second if $\lambda < 0$.

These simple remarks enable us to study the form of the plane sections of a real cubic surface, which, by virtue of § 30, we can suppose to be on the point of degenerating into 3 planes. In the next section we shall deal with this in another way: here we merely hint at the possibility, by considering a convenient pencil of plane sections of a degenerating surface, of showing that:

While the real cubic surfaces of the types F_1, F_2, F_3, F_4 consist (in the projective real field) of a single connected piece, a surface F_5 consists of two different pieces, one of which is ovoidal and contains three ∞^1 systems of conics, intersected on it by the planes through one of the three real lines of F_5. When F_5 degenerates into three independent planes, of which one is necessarily real and the other two are conjugate complex, the ovoidal piece tends to a segment situated upon the intersection of the two conjugate complex planes, while the other piece tends to the real plane and to another segment of the intersection of the complex planes, having one extremity upon that plane.

Thus, for instance, the Cartesian equation

$$(x+y+z)(x^2+y^2)+\lambda(z-1)(z-2)(z-3) = 0,$$

with λ real, positive, and sufficiently small, defines a surface F_5 (§ 30), which—in the projective real field—obviously consists of an ovoidal

piece belonging to the region $2 \leqslant z \leqslant 3$ and of another odd piece; when $\lambda \to 0$, the former piece tends to the segment $2 \leqslant z \leqslant 3$ of the z-axis, and the latter tends to the plane $x+y+z = 0$ plus the segment $0 \leqslant z \leqslant 1$ of the z-axis.

64. A real cubic surface of the type F_1, F_2, F_3, or F_4, or the non-ovoidal piece of a surface F_5, is an irreducible closed 2-dimensional manifold, on which all the real lines of the surface lie. We shall see that the manifold itself is divided by these lines into a certain number of cells, constituting what we call the *generalized polyhedron* of the cubic surface. If a non-singular cubic surface is deformed continuously so that it remains non-singular during the deformation, its generalized polyhedron is also submitted to a continuous deformation, which, however, may not be an isomorphism, since some triangular face of the polyhedron disappears when three real coplanar lines of the surface come to have a point in common, that is, if the surface acquires an Eckardt point of the 1st type (§ 101).

In the remainder of this section we study the generalized polyhedron of a cubic surface F_1, F_2, F_3, F_4, or F_5, in the general case in which the surface has no Eckardt points of the 1st type. But, by means of limiting considerations, it is very easy to see what happens to the generalized polyhedron in the case in which the surface acquires one or more such points; and the classification of the non-singular cubic surfaces having this peculiarity is accomplished below, in section XIV.†

If a non-singular cubic surface F (having no Eckardt points of the 1st type) contains l real lines, each of which is incident with i of the remaining real lines, the generalized polyhedron of F has evidently

$$s = li$$

sides (each containing 2 vertices and common to 2 faces), and

$$v = \tfrac{1}{2}li$$

vertices (each belonging to 4 sides and common to 4 faces).‡ We recall, moreover, that the expression

$$c = -v+s-f,$$

where f is the number of the *faces*, is a topological invariant of the polyhedron, called its *characteristic*; if it is odd, the surface covered by

† The incidence matrices of such polyhedra could easily be derived from our next investigation; this we leave to the reader.

‡ These points are among those we have called k-points in § 9.

the polyhedron is certainly *non-orientable* and topologically characterized by its *genus*

$$p = c+2.\dagger$$

The sides of the polyhedron are said to be *elliptic* or *hyperbolic*, according to the sort of line of F to which they belong; and two distinct sides having a vertex V in common are said to be *opposite* or *adjoining* at V, according as they belong to the same line or to two distinct lines crossing in V. It is clear that *two consecutive sides of the same face are adjoining at their common vertex*; in addition we have that:

Of two sides adjoining another side at its two vertices one is always elliptic and the other hyperbolic, unless all three sides are hyperbolic.

This result is an immediate consequence of §§ 35, 36, in the case of the cubic surfaces of the type F_1; and it can be established without difficulty also in the other cases, by means of our graphical representation. From this it follows that:

Every face of the generalized polyhedron of a cubic surface contains (*at least*) *two adjoining hyperbolic sides.*

We now proceed to examine separately the generalized polyhedra of the different types of cubic surfaces.

65. Let us consider, on a cubic surface F_1, two hyperbolic lines r, s having a point T in common; F_1 has then the plane rs as a tritangent plane of the 1st kind: and we have two essentially different cases, according as this tritangent plane is or is not principal (§ 36).

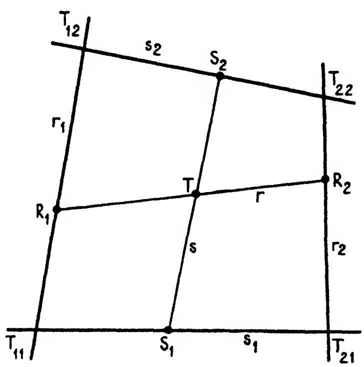


FIG. 59

In the first case, denoting by TR_1, TR_2 and by TS_1, TS_2 the two pairs of opposite sides having T as vertex and belonging to r and s respectively, we shall prove that

Upon F_1, each of the (*elliptic*) *lines r_1, r_2 incident with r in R_1, R_2 is incident with each of the* (*elliptic*) *lines s_1, s_2 incident with s in S_1, S_2; the faces of the generalized polyhedron having T as vertex are 4 quadrangular cells, whose sides determine the skew quadrilaterals $s\,r\,r_i\,s_j$ $(i, j = 1, 2)$, and together constitute a single quadrangular cell* (cf. Fig. 59).

† Cf., for instance, H. Seifert—W. Threlfall, *Lehrbuch der Topologie* (Leipzig, Teubner, 1934), §§ 38, 39.

By virtue of §§ 36, 38, we can, in fact, without restriction suppose that

$$r = 202, \qquad s = 022;$$

then we have

$$r_1, r_2 = 012, 032, \qquad s_1, s_2 = 102, 302,$$

so that r_i and s_j are incident in a point, T_{ij} say, and it is possible to let F_1 degenerate into three planes in such a way that all the points T, R_1, R_2, S_1, S_2, T_{11}, T_{12}, T_{21}, T_{22} have the same limit P_{32}; whence the results stated follow at once.

In the case in which the tritangent plane rs is non-principal, we denote by t the third line of F_1 belonging to it, and by R, S the points

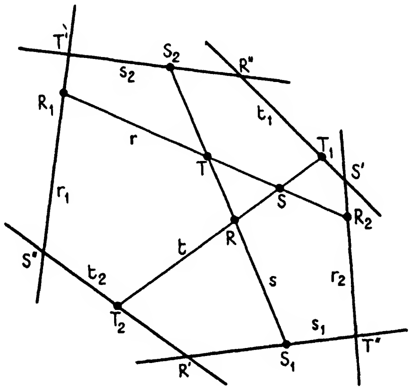


FIG. 60

of intersection of this line with s, r respectively. One of the two segments determined on t by R, S is a side RS of the generalized polyhedron of F_1 (§ 36), whose opposite sides at S, R we indicate by ST_1, RT_2 respectively; we denote by t_1 and t_2 the (elliptic) lines of F_1 incident with t at T_1, T_2, and introduce a similar notation for the lines r, s, as shown in Fig. 60. Then we can say that

$r_1 s_1 t_1$ and $r_2 s_2 t_2$ are two complementary triplets of elliptic lines of the 1st type. There is a skew hexagon $R'S''T'R''S'T''$ determined by the lines $r_1 s_2 t_1 r_2 s_1 t_2$ taken in this order, each side of which consists of two opposite sides of the generalized polyhedron inherent to F_1. Such a hexagon is the contour of a hexagonal cell of F_1, which is divided by r, s, t into 7 cells; these cells are the faces of the polyhedron which have at least one of the points R, S, T as a vertex: one of them is triangular, three are quadrangular, and the other three are pentagonal, as shown in Fig. 60.

By virtue of §§ 36, 38, we can, in fact, without restriction suppose

$$r = 310, \qquad s = 130, \qquad t = 220;$$

then, from the graphical representation (Fig. 61), it follows that

$$r_1 = 230, \qquad r_2 = 120, \qquad s_1 = 210, \qquad s_2 = 320$$

and

$$\text{either } t_1 = 021, \qquad t_2 = 201, \quad \text{or} \quad t_1 = 023, \qquad t_2 = 203,\dagger$$

so that the above stated incidences among these lines really occur.

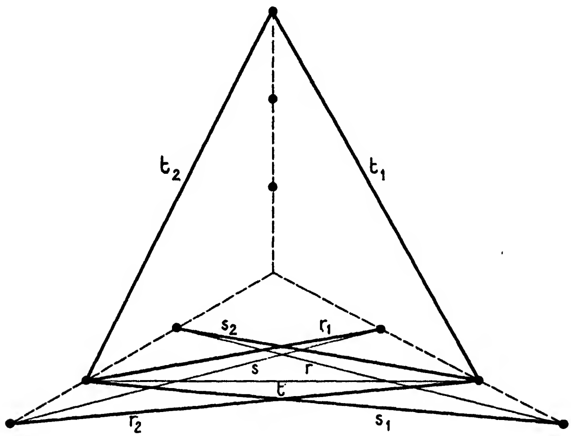


FIG. 61

The other results enunciated are a simple consequence of the fact that it is possible to let F_1 degenerate into three planes, in such a way that—in the limit—the two sides of the skew hexagon $R'S''T'R''S'T''$ situated on t_1, t_2 vanish, and the lines r, s, t, r_1, r_2, s_1, s_2 all come to belong to one of these three planes, in the manner represented by Fig. 61.

This figure, compared with Fig. 60, shows that the 5 tritangent planes

$$\alpha_1 = s_2 r_1 r_2, \qquad \alpha_2 = s_2 s s_1, \qquad \alpha_3 = s_2 t_1 = (321),$$
$$\alpha_4 = (322), \qquad \alpha_5 = (323)$$

through $s_2 = 320$, occur in their pencil in the cyclic order $\alpha_1 \alpha_2 \alpha_3 \alpha_4 \alpha_5$. We can therefore, adopting the notation of § 59, represent the elliptic lines of F_1 in the manner indicated by Fig. 62, so that the complementary triplets $r_1 s_1 t_1$ and $r_2 s_2 t_2$ become $a_1 a_2 a_3$ and $b_4 b_6 b_5$ respectively,

† In Fig. 61 the first of these two alternatives is represented.

and from § 59 it follows that they are of the 1st type. From this and from §§ 36, 59 we deduce without difficulty that:

Each of the 10 *pairs of complementary triplets of elliptic lines of the* 1*st type, and each of the* 10 *pairs of complementary triplets of elliptic lines of the* 2*nd type, determines as follows one of the* 10 *non-principal tritangent*

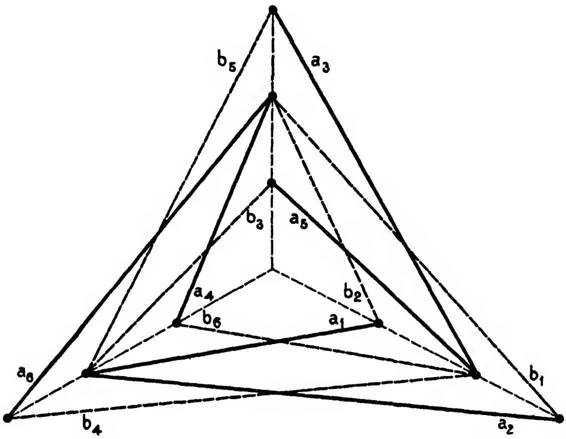


FIG. 62

planes of the 1*st kind: the* 2 *triplets of the pair considered are related in one-to-one correspondence, and the further lines of* F_1 *which belong to the* 3 *planes joining their corresponding lines are the lines* r, s, t *of a non-principal tritangent plane of the* 1*st kind. According as the pair is of the* 1*st or* 2*nd type, these* 3 *planes are the tritangent planes of the* 2*nd kind containing one of the lines* r, s, t *which are consecutive or, respectively, non-consecutive to the plane rst; thus, if the plane rst is given, we have the means of constructing the* 2 *pairs of complementary triplets associated to it, which therefore are mutually residual.*

With the notation of § 64, for a surface F_1 we have $l = 27$, $i = 10$, and therefore

$$v = 135, \qquad s = 270.$$

Moreover, in correspondence with the 5 principal planes of F_1 we obtain $3.5 = 15$ sets of 4 quadrangular faces of the type represented by Fig. 59; and, in correspondence with the 10 tritangent planes of the 1st kind which are not principal, we have 10 sets of 7 faces of the type represented by Fig. 60. Since the faces thus enumerated are all distinct and, by virtue of the final remark of § 64, exhaust the generalized polyhedron inherent to F_1, we have that:

The generalized polyhedron of F_1 *contains in all*

$$f = 130$$

faces; 10 *of these are triangular,* 90 *are quadrangular, and* 30 *are pentagonal cells. The* 90 *quadrangular faces are of* 2 *different types:* 30 *of them (which we call of the* 1*st type) have a vertex in common with a triangular face, while each of the other* 60 *(which we call of the* 2*nd type) has no point in common with any triangular face.*

The characteristic of such a polyhedron being $c = 5$ (§ 64), we have that:

A cubic surface F_1 *consists of a single* 2*-dimensional closed manifold, which is non-orientable and of genus* $p = 7$.

The following remarks throw light on the structure of the non-orientable polyhedron inherent to F_1. Let us consider any one of the 15 hyperbolic lines of F_1, r say; through it we have 2 non-principal tritangent planes of the 1st kind (§ 36), each containing two further lines of F_1, say s, t and s^*, t^*. We associate with the triangle rst the configuration represented by Fig. 60 and with the triangle rs^*t^* the analogous configuration, for which we adopt a similar notation with starred letters; rr_1r_2 and $rr_1^*r_2^*$ are therefore the 2 tritangent planes of the 2nd kind through r: the fifth tritangent plane through this line is principal of the 1st kind, and we denote by q_1, q_2 its further 2 lines and by Q_1, Q_2 the points rq_1, rq_2. By virtue of § 36 we can suppose the 10 points $T^*\ R_1^*\ Q_1\ R_1\ T\ S\ R_2\ Q_2\ R_2^*\ S^*$ occurring on r in this cyclic order; then, by applying the first result of this paragraph to the pairs of hyperbolic lines r, q_1 and r, q_2, we see that we must have

$$\text{either}\quad s_1 = s_1^*, \qquad t_1 = t_1^* \quad \text{or} \quad s_1 = t_1^*, \qquad t_1 = s_1^*,$$

and

$$\text{either}\quad s_2 = s_2^*, \qquad t_2 = t_2^* \quad \text{or} \quad s_2 = t_2^*, \qquad t_2 = s_2^*.$$

As the hexagons determined by $r_1s_2t_1r_2s_1t_2$ and by $r_1^*s_2^*t_1^*r_2^*s_1^*t_2^*$ cannot have any vertex in common, the points s_1t_2, s_2t_1 must be distinct from $s_1^*t_2^*$, $s_2^*t_1^*$, so that:

$$\begin{array}{lllll}\text{either} & s_1 = s_1^*, & t_1 = t_1^*, & s_2 = t_2^*, & t_2 = s_2^*,\\ \text{or} & s_1 = t_1^*, & t_1 = s_1^*, & s_2 = s_2^*, & t_2 = t_2^*.\end{array}$$

We therefore see that the 2 hexagonal cells of F_1 having these hexalaterals as contours, and the 2 quadrangular cells of F_1 having as contours $r_1s_2r_1^*t_2$ and $r_2s_1r_2^*t_1$, constitute together a Möbius strip having the quadrilateral $s_1t_2t_1s_2$ as contour; this Möbius strip consists of $2.7+2.4 = 22$ faces of the generalized polyhedron inherent to F_1,

which are precisely the 20 faces having one side on r and two quadrangular faces of the 2nd type, opposite in R, R^* to those RST, $R^*S^*T^*$ of the former which are triangular. We can say in conclusion that:

A cubic surface F_1 is divided by its 12 elliptic lines into 15 quadrangular cells (each consisting of 4 quadrangular faces of the 2nd type of its polyhedron) and 10 hexagonal cells (each consisting of 1 triangular, 3 quadrangular of the 1st type, and 3 pentagonal faces of the polyhedron); any 2 adjacent ones of these 25 cells are one quadrangular and the other hexagonal, and belong to a well-defined cycle of 4 cells each of which is adjacent to 2 other cells of the set. The 4 cells of such a cycle constitute a Möbius strip, containing in the interior one of the hyperbolic lines of F_1; thus we have 15 Möbius strips, in correspondence with the 15 hyperbolic lines of F_1: and the mutual relations of the former are connected in an obvious manner with the incidences of the latter.

66. Taking into account § 42, and by means of considerations quite similar to those developed in § 65 for the surfaces F_1, we can establish the following theorem, which completes the discussion of § 64 for the surfaces F_2:

A cubic surface F_2 contains 9 hyperbolic lines, which form a Steiner set and therefore determine the 6 planes of a Steiner trihedral pair. In each of these 6 planes we have 3 of these lines, intersecting 2 by 2 in 3 points; the faces of the generalized polyhedron of F_2 having at least one of these points as vertex are 7 cells (1 triangular, 3 quadrangular, and 3 pentagonal) and together make up a single hexagonal cell having as contour a hexalateral, whose sides belong to the 6 elliptic lines of F_2 (cf. Fig. 60). The polyhedron itself is simply the sum of the 6 sets of 7 faces that we thus obtain in correspondence with these 6 tritangent planes (of the 1st kind); so that its characters are

$$l = 15, \qquad i = 6, \qquad v = 45, \qquad s = 90, \qquad f = 42, \qquad c = 3$$

and F_2 is a unilateral (closed, irreducible) manifold of genus $p = 5$.

The 6 hexagonal cells into which F_2 is divided by its 6 elliptic lines are such that any 2 of them having a side in common have also in common another side, opposite to the former in both their contours; together they constitute a Möbius strip, which has as contour a skew quadrilateral and contains in its interior one of the hyperbolic lines of F_2. There are in all 9 such Möbius strips; and, in correspondence with the 6 triplets of hyperbolic lines of F_2, we have 6 different possibilities of considering F_2 as the sum of 3 of these Möbius strips.

Two hexagonal cells of F_2 adjacent to a third hexagonal cell are never mutually adjacent. The 6 hexagonal cells can (in a single way) be distributed in 2 sets of 3 cells, such that 2 of them are or are not adjacent according as they respectively belong to different sets or to the same set.†

67. Let us consider in the projective space a real conic $\mathfrak{C} = \mathfrak{C}(t)$, which is a continuous function of t in an interval $t_1 \leqslant t \leqslant t_2$, and such that $\mathfrak{C}(t')$ and $\mathfrak{C}(t'')$ never have any point in common if $t' \neq t''$. The surface described by $\mathfrak{C}(t)$ is clearly homeomorphic to an annulus if no $\mathfrak{C}(t)$ is singular, and to a circle if that is true save for $\mathfrak{C}(t_1)$ or $\mathfrak{C}(t_2)$ which (in the real domain) reduces to a single point; if $\mathfrak{C}(t)$ is non-singular for $t_1 < t < t_2$, but $\mathfrak{C}(t_1)$ and $\mathfrak{C}(t_2)$ both degenerate into two (real) distinct lines or one degenerates into two distinct lines and the other is reduced to a single point, then the above-considered annulus or circle has to be replaced by the surface derivable from it by identifying a pair of distinct points upon each circumference of the contour.

The previous remarks lead to a simple determination of the generalized polyhedron inherent to a cubic surface F_3. Such a surface, in fact, contains a hyperbolic line of the 2nd kind, r, through which there are five real tritangent planes; two, α, γ, of these are of the 1st kind, one, β, is of the 2nd kind, the other two, δ, ϵ, being of the 3rd kind (§ 31, iii); and we can suppose the planes $\alpha\ \beta\ \gamma\ \delta\ \epsilon$ to occur in this order in their pencil. Let us denote by $a_1 a_2$, $b_1 b_2$, $c_1 c_2$ the pairs of real lines which are the further intersections of F_3 with α, β, γ respectively, by $A_1 A_2$, $B_1 B_2$, $C_1 C_2$ the three pairs of points (of an involution) intersected by them on r, and by A, B, C the points common to such pairs of lines; we can, for instance, suppose $A_1 B_1 C_1 C_2 B_2 A_2$ to occur on r in this cyclic arrangement. The planes through r which intersect F_3 further are those belonging to the angle $\delta\epsilon$ which contains α, β, and γ; the further intersection of such a plane ρ is a real conic $\mathfrak{C}$, which is non-singular, save when ρ coincides with one of the planes α, β, γ or δ, ϵ, in which cases $\mathfrak{C}$ is given by the pairs of lines $a_1 a_2$, $b_1 b_2$, $c_1 c_2$, or reduces to a single point respectively.

Then we see that each of the loci generated by $\mathfrak{C}$ when ρ describes one of the angles $\alpha\beta$, $\beta\gamma$, $\gamma\delta$, $\epsilon\alpha$ is divided by r into two cells, having the same vertices, which are $A_1 B_1 B B_2 A_2 A$, $B_1 C_1 C C_2 B_2 B$, $C_1 C C_2$,

† The distribution of the 6 hexagonal cells into two such sets corresponds to the distribution of the 6 tritangent planes of the 1st kind of F_2 into two conjugate Steiner trihedra, and implies the possibility of *drawing a coloured map of the generalized polyhedron determined on F_2 by its 6 elliptic lines, with the use of only two colours.*

A_1AA_2 respectively. Since $r\ a_1\ a_2\ b_1\ b_2\ c_1\ c_2$ are the only real lines of F_3, we obtain the result:

The generalized polyhedron of F_3 has 9 *vertices,* 18 *sides, and* 8 *faces* (4 *triangular and* 4 *hexagonal*)*; its characteristic is therefore $c = 1$, so that F_3 is unilateral and of genus $p = 3$.*

68. A cubic surface F_4, or the non-oval piece of a cubic surface F_5, can be continuously deformed into a real plane and a segment having one extremity on it (§§ 30, 63); since the 3 real lines of the cubic surface have as limit 3 lines dividing the real plane into 4 triangular regions, it follows that:

The generalized polyhedron of F_4 or F_3 has 3 *vertices,* 6 *sides, and* 4 *triangular faces, and is homeomorphic with the projective plane.*

XII. The parabolic curve of the real non-singular cubic surfaces, and the classification of the latter in the affine space

69. Let us consider the *dual surface* of a real non-singular cubic surface F, namely, the real surface $\hat{F}$ (of order 12) transform of F by means of a reciprocity. By associating with each point P of F the point $\hat{P}$ of $\hat{F}$ which corresponds by means of this reciprocity to the plane touching F at P, we obtain a one-to-one correspondence between F and $\hat{F}$, having no exceptional point on F, which transforms each line r of F into a double line $\hat{r}$ of $\hat{F}$, and each point common to two lines of F into a triple point of $\hat{F}$ common to three double lines of this surface; a point of $\hat{r}$ corresponds to two points of r, which are conjugate in the involution defined by F upon this line (§ 6). It is, moreover, well known that to the asymptotic curves and the parabolic curve of F correspond the asymptotic curves and the cuspidal curve of $\hat{F}$.

In the real field, consequently, we have that an elliptic or hyperbolic point of $\hat{F}$ corresponds to a point of F which is still elliptic or hyperbolic. A real line of $\hat{F}$ corresponding to an elliptic line of F is a locus of nodal double points of $\hat{F}$, with the exception only of some triple point; on the contrary, a line $\hat{r}$ of $\hat{F}$ corresponding to a hyperbolic line r of F contains two distinct cuspidal points, which correspond to the parabolic points of r and divide $\hat{r}$ into two segments, one of which is nodal for $\hat{F}$, while the other is isolated. The lines of $\hat{F}$ divide the non-oval sheet of this surface into a certain number of (possibly singular) cells, each of which corresponds to a cell of the generalized polyhedron of F, and has no singular point in the interior; such cells constitute what we call the *generalized polyhedron of $\hat{F}$*. The study of

the form of the surfaces F, $\hat{F}$ is closely connected with some properties of their generalized polyhedra which we propose to explain. We have particularly to consider the connected *regions* into which the surface $\hat{F}$ divides the space in which it is embedded. Each of these regions has as contour a 2-dimensional cycle of $\hat{F}$, and its interior corresponds to a *continuous* ∞^3 *system of non-singular plane sections of* F. It is obvious that two plane sections of F belonging to the same system have the same form and can be connected by a continuous chain of non-singular plane sections of F; this is no longer true for two non-singular plane sections belonging to different systems. The determination of these continuous systems gives at once the classification of the surfaces homographic with F from the *affine* point of view.

In the following paragraphs we shall study these subjects, beginning with some general remarks and then examining separately the different cases which F and $\hat{F}$ may present. Now we confine ourselves to pointing out that the non-singular plane sections of any one of the above continuous systems either all consist of an oval and an odd branch, or all consist of a single odd branch; we distinguish the two cases by saying that the continuous system, as well as the corresponding region determined by $\hat{F}$, is *of the* 1*st* or, respectively, *of the* 2*nd kind*. By virtue of § 63 we have that:

Two adjacent regions, i.e. two distinct regions whose contours have a 2*-dimensional portion in common, are always of different kinds.*

70. Let us consider a parabolic point of any surface F, i.e. a simple point M of F in which the section $\mathfrak{C}$ of F with the plane μ touching F at M has (at least) a cuspidal point.

If $\mathfrak{C}$ *has in* M *a point of multiplicity* 2 (*and not greater*), *the necessary and sufficient condition that the line touching* $\mathfrak{C}$ *at* M *has in this point at least two intersections with the parabolic curve* $\mathfrak{L}$ *of* F, *is that* M *is a tacnodal point of* $\mathfrak{C}$; *in this case, if* M *is also an inflexion for one of the two branches of* $\mathfrak{C}$ *through it, the other branch and* $\mathfrak{L}$ *have in* M *a symmetric contact. The necessary and sufficient condition that* $\mathfrak{L}$ *has in* M *a point of multiplicity* $\geqslant 2$, *is that* $\mathfrak{C}$ *has in that point a symmetric tacnode.*†

† Two tangent linear branches of the xy-plane, of equations $y = hx^2+\dots$, $y = kx^2+\dots$, have the projective tac-invariant h/k, of which C. Segre first gave a projective interpretation (cf. C. Segre, 'Su alcuni punti singolari delle curve algebriche, e sulla linea parabolica di una superficie', *Rendic. R. Acc. Naz. Lincei*, (V) vol. 6 (1897)$_2$, pp. 168–75, and also B. Segre, 'Sui sistemi continui di curve piane con tacnodo', ibid., (VI) vol. 9 (1929)$_1$, pp. 970–4; for further extensions, cf. B. Segre, 'Sugli elementi curvilinei che hanno comuni le origini ed i relativi spazi osculatori', ibid., (VI) vol. 22 (1935)$_2$, pp. 392–9). When this invariant has the value -1, the two branches have a *symmetric contact* (namely, each of them osculates the symmetric of the other with respect to the

To see this, we introduce non-homogeneous projective coordinates x, y, z, all vanishing at M, and represent F in the neighbourhood of this point by an equation of the type $z = f(x, y)$, with

$$f(x,y) \equiv y^2+(ax^3+bx^2y+cxy^2+dy^3)+ex^4+\dots .$$

Then the curve $\mathfrak{C}$ and the curve $\mathfrak{L}'$ projection of $\mathfrak{L}$ on μ from the point at infinity of the z-axis, have the equations $z = 0, f(x,y) = 0$, and

$$z = 0, \qquad \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2}-\left(\frac{\partial^2 f}{\partial x \partial y}\right)^2$$
$$\equiv (6ax+2by+12ex^2+\dots)(2+\dots)-(2bx+2cy+\dots)^2 = 0.$$

The tangent line of $\mathfrak{C}$ at M is the x-axis; and this line has in M multiplicity of intersection $\geqslant 2$ with $\mathfrak{L}'$ if, and only if, $a = 0$, i.e. if $\mathfrak{C}$ has in M a tacnodal point. In this case, $e = 0$ is the further condition that (at least) one of the two branches of $\mathfrak{C}$ through M has an inflexion at M; the other branch has then the equations $z = 0$, $y = -bx^2+\dots$, and the equations of $\mathfrak{L}'$ are $z = 0$, $y = bx^2+\dots$, so that the projective tac-invariant of these two curves at M is -1, i.e. they have there a symmetric contact. Finally, the conditions for $\mathfrak{L}$ to have in M a point of multiplicity $\geqslant 2$ are simply $a = b = 0$, which are equivalent to the conditions that $\mathfrak{C}$ has in M a symmetric tacnode.

It is at once seen that:

If $\mathfrak{C}$ has in M a triple point, with three non-coincident tangents, the parabolic curve $\mathfrak{L}$ has in M a double point, and the two tangents of $\mathfrak{L}$ at M constitute the Hessian of the three lines. In particular, in the real domain, we have that M is an isolated double point or a nodal point of $\mathfrak{L}$, according as the three tangents of $\mathfrak{C}$ at M are distinct and all real, or one real and two conjugate complex.

71. No plane section of a non-singular cubic surface can have a symmetric tacnode, since a cubic curve with a symmetric tacnode contains as a component the tacnodal tangent counted twice. Therefore from § 70 it follows that:

The parabolic curve $\mathfrak{L}$ of any non-singular cubic surface F is generally non-singular, and can have at most a finite number of ordinary double points; the real singular points of $\mathfrak{L}$ are the real Eckardt points of F, each of which is an isolated or a nodal double point of $\mathfrak{L}$, according as it is an intersection of three real lines or of two conjugate complex and one

common tangent line), and constitute together a *symmetric tacnode*; this notion goes back to E. Wölfing, 'Ueber die Hesse'sche Covariante einer ganzen rationalen Funktion von ternären Formen', *Math. Ann.*, vol. 36 (1890), pp. 97–120, § 8. The last part of the above theorem has already been proved in the § 3 of C. Segre's paper just quoted.

real line of F. The curve $\mathfrak{L}$ touches each of the 27 lines of F in its two parabolic points, having in each of them a symmetric contact with the conic which, together with that line, constitutes the intersection of F with the parabolic plane defined by such a point.

72. If M is an arbitrary hyperbolic point of a non-singular cubic surface F, not lying on any line of F, the curve $\mathfrak{C}$ of intersection of F with the plane μ touching F at M is irreducible and has a nodal point at M; the point M divides $\mathfrak{C}$ into two (in the projective sense) closed simple branches, of which one—which we call the *loop* of $\mathfrak{C}$—is even, while the other is odd. There is a certain number (possibly zero) of real lines of F intersecting μ in points (certainly distinct from M) of the loop of $\mathfrak{C}$; and the set formed by these lines we call the *set of lines inherent to M*. A similar set can also be considered in the case in which M is an elliptic or parabolic point of F; in such a point the loop of the section with the tangent plane reduces to the point itself, so that we can say that the set of lines inherent to it is the null-set. It is clear that, when the point M moves continuously on F, without crossing any line of F, the set of lines inherent to it remains always the same. It follows that

All the points interior to any given face of the generalized polyhedron of F are inherent to the same set of lines, which we call the set of lines inherent to that face.

Let us now consider on F a non-parabolic point N of a real line r, which, moreover, does not lie on any other line of F. The tangent plane at this point meets F in r and an irreducible conic $\mathfrak{D}$, intersecting r in N and in a further point N'. When the point M tends to N, the curve $\mathfrak{C}$ breaks up into $r+\mathfrak{D}$, and the loop of $\mathfrak{C}$ may *a priori* only tend to the closed curve formed by the well-defined segment NN' which is interior to the conic $\mathfrak{D}$ and by one or the other of the two arcs of this conic having N, N' as extremes. By considering the behaviour of $\mathfrak{C}$ in the neighbourhood of N' and remembering § 63, we see that both these cases are possible; and that the loop of $\mathfrak{C}$ has one or the other limit, according as the point M reaches N from one or the other side of r.

The real lines of F other than r can be distributed into two sets, which we call $\mathfrak{I}_r$ and $\mathfrak{H}_r$ respectively, consisting of the lines which are incident or skew with r. Taking into account that the former are skew to $\mathfrak{D}$ and the latter are incident to $\mathfrak{D}$, and that a line of $\mathfrak{I}_r+\mathfrak{H}_r$ incident with the loop of $\mathfrak{C}$ is also incident with its limit and conversely, we see that

Two faces of the generalized polyhedron of F adjacent along a line r have as inherent lines the same lines of $\mathfrak{J}_r$; on the contrary, each line of $\mathfrak{H}_r$ is inherent to one and only one of the two faces.

Let us now choose on F a conic Δ distinct from $\mathfrak{D}$, intersecting r in two distinct points P, P'; on r the pairs N, N' and P, P' separate or do not separate each other, according as r is an elliptic or hyperbolic line of F (§ 27). When $\mathfrak{C}$ is sufficiently near to $r+\mathfrak{D}$, it has exactly two intersections with Δ, which are respectively near to P and P'. It follows that, if r is elliptic, one and only one of these intersections belongs to the loop of $\mathfrak{C}$, so that the plane of $r+\Delta$ must have a further intersection with this loop, necessarily situated on r; if, on the contrary, r is hyperbolic, neither or both of the former two intersections belong to the loop of $\mathfrak{C}$, and r does not intersect this loop. Hence:

The sides of any given face of the generalized polyhedron of F which are inherent to the face itself are the elliptic ones, and only these.

A face of the generalized polyhedron having as inherent the null-set can have only hyperbolic sides, so that (section XI) it must be triangular; hence

The parabolic curve of F lies entirely within the triangular faces of the generalized polyhedron of F; all the points of a non-triangular face are hyperbolic, excepting only (if they exist) the parabolic points on its sides, which, of course, are parabolic points of the surface.

We shall see later on that, conversely, each triangular face contains just one branch or loop of the parabolic curve.

(i) The surfaces F_1 and $\hat{F}_1$

73. The generalized polyhedron of a cubic surface F_1 has 10 triangular faces, which are 2 by 2 non-adjacent; and every side of each of these faces contains in the interior one parabolic point, through which the parabolic curve $\mathfrak{L}$ of F goes simply, touching that side in it (§§ 65, 71). By virtue of the last theorem of § 72, each triangular face must therefore contain at least one branch of $\mathfrak{L}$. This curve, on the other hand, can have as singular points only isolated double points, such points being precisely the Eckardt points of F_1 (§ 71). If we consider a triangular face of F_1, we can—by deforming F_1 continuously and keeping it non-singular—transform the 3 lines determining its contour into 3 lines concurring in a point, which then is an Eckardt point of the surface derived from F_1; for reasons of continuity (similar to those indicated in § 63), the portion of $\mathfrak{L}$ belonging to the triangular face is continuously deformed into the above Eckardt point, and becomes a

small oval near this point a little before reaching its final position. It follows that it consists of a single closed branch, so that

The parabolic curve of a surface F_1 without Eckardt points consists of 10 *distinct non-singular branches; each of these branches belongs to one of the* 10 *triangular faces of the generalized polyhedron of F_1, being inscribed in its triangular contour, and enclosing a* 2*-dimensional cell of elliptic points of F_1. Thus there are on F_1* 10 *distinct simply connected regions of elliptic points, and a single connected region of hyperbolic points having as contour the* 10 *branches of the parabolic curve.*†

As a consequence, taking into account §§ 65, 71, 72, we obtain that:

If an irreducible conic of F_1 touches one of the lines of this surface, the neighbourhood of the point of contact upon the conic belongs to a pentagonal face of the generalized polyhedron of F_1, and the conic itself intersects the two elliptic sides of this face.

We see, moreover, that the hyperbolic or parabolic points of a triangular face which do not belong to its contour constitute three open 2- or, respectively, 1-dimensional cells.

74. In virtue of §§ 72, 73, we have that:

The 10 *triangular faces of the generalized polyhedron of F_1, and only these, have the null-set as their inherent set of lines.*

This theorem marks a distinction, in relation to the lines of F_1, between the triangular and the non-triangular faces of the generalized polyhedron; but it does not show whether a similar distinction also exists among the 10 triangular faces themselves. We complete this point by means of the following considerations, important also for further deductions.

Let us consider on F_1 one of the two sextuplets of elliptic lines and an irreducible plane section $\mathfrak{C}$, whose oval or loop (if it exists) meets none of the 27 lines of F_1. Then the 6 lines a_i ($i = 1, 2, \ldots, 6$) of that sextuplet meet $\mathfrak{C}$ in 6 distinct points C_i, which occur on $\mathfrak{C}$ in a well-defined cyclic arrangement, $C_1 C_2 C_3 C_4 C_5 C_6$ say. Each of the 60 transformations of the group Γ_1 induces a certain substitution among the 6 lines, and therefore also among the 6 points, which is sufficient to determine it completely (§ 38); we shall show that:

† In accordance with § 64, we omit to state explicitly what modifications occur if F_1 has Eckardt points. An Eckardt point of F_1 is a parabolic point for each of the 3 lines of F_1 which concur in it, the other parabolic points being 3 collinear points (§ 6); by reasons of continuity and using again § 6, it follows that:

The points of contact of any branch of the parabolic curve with the sides of the triangular face in which it is inscribed, joined to the opposite vertices, give 3 lines of a pencil.

A similar result also holds for the cubic surfaces of the other types.

There are 6 *transformations of* Γ_1 *which do not change the cyclic arrangement* $a_1 a_2 a_3 a_4 a_5 a_6$, *constituting a sub-group simply isomorphic with a symmetric group of degree* 3; *so that there are* 10 *distinct cyclic arrangements induced among those* 6 *lines by the* 60 *transformations of* Γ_1.

In fact, the substitutions among the 6 lines a_i which do not alter their cyclic arrangement are the cyclic substitution of order 6

$$\rho = (a_1 a_2 a_3 a_4 a_5 a_6)$$

and their powers, and the involutory transformations

$$\sigma = (a_2 a_6)(a_3 a_5), \qquad \sigma' = (a_3 a_1)(a_4 a_6), \qquad \sigma'' = (a_4 a_2)(a_5 a_1),$$
$$\tau = (a_2 a_1)(a_3 a_6)(a_4 a_5), \qquad \tau' = (a_3 a_2)(a_4 a_1)(a_5 a_6),$$
$$\tau'' = (a_4 a_3)(a_5 a_2)(a_6 a_1),$$

which all together constitute a group of order 12. No transformation of Γ_1 can induce among the lines a_i an odd power of ρ, since otherwise a convenient power of the former would induce the substitution ρ^3 among the latter, which changes the triplet $a_1 a_2 a_3$ into $a_4 a_5 a_6$: and this is impossible, as these 2 residual triplets are always of different types (§ 59). The three substitutions τ, τ', τ'' must likewise be excluded, as each of them interchanges 2 residual triplets; so that the only substitutions among the lines a_i which can be induced by a transformation of Γ_1 which preserves their cyclic arrangement are the identity and the substitutions ρ^2, ρ^4, σ, σ', σ'', which obviously form a *group simply isomorphic with a symmetric group of degree* 3. The fact that each of these substitutions is really induced by a transformation of Γ_1 could be proved without difficulty by means of § 38 and our graphical representation; but it also follows simply by remarking that, if this were not so, we should have more than 10 distinct cyclic arrangements induced among the 6 lines a_i by the 60 transformations of Γ_1, which is incompatible with one of the next developments.

If, in particular, $\mathfrak{C}$ is the section of F_1 with a plane γ touching this surface at a point C interior to one of the 10 triangular faces of its generalized polyhedron, the 6 points C_i occur on $\mathfrak{C}$ in an order inducing a certain cyclic arrangement among the 6 lines a_i; *this cyclic arrangement does not change by interchanging C with a point of the same triangular face, but it certainly changes if instead of C we consider a point of another triangular face.* In fact, a transformation of Γ_1 altering the cyclic arrangement induced by $\mathfrak{C}$ among the lines a_i can be performed by means of a convenient circulation of F_1; if we follow continuously the corresponding deformation of $\mathfrak{C}$, we obtain that, at the end of the

circulation, the plane of the curve deduced from $\mathfrak{C}$ must touch F_1 at a point of another triangular face. In such a way 10 (certainly existing) transformations of Γ_1, inducing among the lines a_i 10 distinct cyclic arrangements, lead to 10 sections of F_1 with planes touching this surface at points interior to 10 different triangular faces of its generalized polyhedron: but the triangular faces are exactly 10 in number, so that any 2 distinct ones among them must induce 2 distinct cyclic arrangements among the 6 lines a_i.

75. In virtue of § 65 we have that each pentagonal face and each quadrangular face of the 1st type of the generalized polyhedron of F_1 has a side or, respectively, a vertex in common with one triangular face of such a polyhedron, and has no points in common with the other triangular faces. Denoting by r the line of F_1 which contains the common side or, respectively, the side of the triangular face opposite to the common vertex, we have that

The set of lines inherent to the pentagonal or quadrangular face initially considered is the set $\mathfrak{H}_r$ consisting of the 16 lines of F_1 skew to r.

In the case of a pentagonal face the fact asserted is an obvious consequence of § 72. The line r, being hyperbolic, cannot in fact belong to the set inherent to that face; and, of the remaining lines of F_1, all those of $\mathfrak{H}_r$ and none of those of $\mathfrak{J}_r$ must belong to such a set, since (§ 74) none of them belongs to the set inherent to the triangular face adjacent along r to the pentagonal face.

In the case of a quadrangular face of the 1st type, we can adopt the notation of § 65 and suppose it to be $RS_1R'T_2$ (Fig. 60). This face is adjacent along the hyperbolic line t to the pentagonal face $RTR_1S''T_2$, which—by virtue of what we have just proved—has as inherent the set $\mathfrak{H}_s$; it follows, again taking into account § 72, that the set of lines inherent to the former quadrangular face is

$$(\mathfrak{J}_t \frown \mathfrak{H}_s)+(\mathfrak{H}_t-\mathfrak{H}_t \frown \mathfrak{H}_s) = \mathfrak{H}_r.$$

76. Let us now consider a quadrangular face of the 2nd type, and one of its two sides belonging to an elliptic line, t_1 say; the face of the generalized polyhedron of F_1 adjacent to the former along this side is either pentagonal or quadrangular of the 1st type (§ 65), so that it has as inherent the set $\mathfrak{H}_s$ of the 16 lines of F_1 skew to a hyperbolic line s, skew to t_1 (§ 75). We have that:

The set of lines of F_1 inherent to the former quadrangular face of the 2nd type is the double-six which contains s and t_1 as corresponding lines.

Such a set, in fact, by virtue of § 72, is simply given by

$$t_1+(\mathfrak{J}_{t_1}\frown\mathfrak{P}_s)+(\mathfrak{P}_{t_1}-\mathfrak{P}_{t_1}\frown\mathfrak{P}_s);$$

but, since s and t_1 are skew, we have

$$\mathfrak{P}_{t_1}-\mathfrak{P}_{t_1}\frown\mathfrak{P}_s = s+(\mathfrak{J}_s\frown\mathfrak{P}_{t_1}),$$

so that the set consists of s, t_1 and the 10 lines of F_1 incident to one and only one of these 2 lines, i.e. (§ 7) it is the double-six indicated above. This is a double-six of the 3rd kind (§ 31, i), since it has as corresponding lines s and t_1, which are one elliptic and the other hyperbolic; with the symbolic notation suggested by Fig. 61, such a double-six can be represented by $\{111\}$, and, taking into account §§ 59, 60, 65, we easily see that it is a *double-six of the 3rd kind and 2nd type*.

In order to investigate the matter further, we suppose (as in § 65) that the 9 lines of F_1 given by Fig. 60 are graphically represented by Fig. 61, and inquire into the exact determination of the 5 principal planes of F_1. For this purpose, we remark that the principal plane through s contains two further lines of F_1, m and n, intersecting s in two points M, N which can be named so that the points MS_2TRS_1N occur on s in this order: then m intersects r_1, t_1 and n intersects r_2, t_2 (§ 65), so that the principal plane smn can only be the plane (133), as is clearly indicated by Fig. 61. In virtue of § 36 we therefore have that the 5 principal planes of F_1 are

(222), (111), (133), (313), (331).

The 2 residual pairs of complementary triplets of elliptic lines, which—in accordance with § 65—are associated with the non-principal tritangent plane rst, are

$$a_1a_2a_3,\ b_4b_6b_5 \quad \text{and} \quad a_4a_6a_5,\ b_1b_2b_3,$$

the notation being the same as in § 65 (the a_i's and b_j's are represented in Fig. 62). The residual intersections of F_1 with the planes joining 2 by 2 (in all the possible manners) the 6 lines of the former or latter pair constitute a Steiner set, containing the lines r, s, t of the tritangent plane initially considered and 6 further lines, which are those of the principal tritangent planes (111) and (331); the 9 lines of this Steiner set are all hyperbolic, that is (§ 32), the Steiner set is of the 3rd kind, and they can be distributed into 3 other tritangent planes of the 1st kind, which are (311), (131), and $\begin{pmatrix}110\\220\\330\end{pmatrix}$, and consequently all non-principal. Since the 10 non-principal tritangent planes of the 1st kind

are equivalent with respect to Γ_1, and F_1 has precisely 10 pairs of *principal planes each of which defines a* Steiner set containing its 6 lines, we can say that:

In the manner indicated above we obtain a one-to-one intrinsic association among the 10 *non-principal tritangent planes of the* 1*st kind, the* 10 *pairs of principal planes, the* 10 *pairs of complementary triplets of elliptic lines of the* 1*st type, the* 10 *pairs of complementary triplets of elliptic lines of the* 2*nd type, and the* 10 *Steiner sets of the* 3*rd kind of* F_1*; any operation of* Γ_1 *which transforms into itself one of those configurations also transforms into itself each of the configurations associated with it.*

We can now return to the quadrangular face of the 2nd type considered at the beginning of this paragraph; Fig. 60 shows that its contour is given by r_2, t_1, t and a line incident with r_2, t and belonging to the principal plane through t, namely (on account of Fig. 61), by the 4 *lines*

$$120 \qquad 021 \qquad 220 \qquad 022. \tag{*}$$

On the other hand, the set {111} inherent to the quadrangular face is the double-six of the 3rd kind containing the *pair of complementary triplets of elliptic lines of the 2nd type*:

$$120,\ 201,\ 012 \qquad 210,\ 102,\ 021; \tag{**}$$

since the Steiner set associated with this pair is given by the 9 lines of the tritangent planes (111), (222), (333), of which the two first are principal and the last is non-principal, it follows that the one-to-one intrinsic correspondence between the two triplets (**) associates the 1st, 2nd, 3rd line of the first triplet respectively with the 1st, 2nd, 3rd line of the second. Hence the relation between the 4 lines (*) and the two triplets (**) can be expressed in the following manner, which—when the 4 lines are given—fully specifies the two triplets, and therefore also the set inherent to the quadrangular face determined by the lines.

Two of the 4 *lines* (*) *are elliptic and two of them are hyperbolic: while the former are two lines of the triplets* (**) *belonging to two different pairs of corresponding lines, the latter are the further intersections of* F_1 *with the planes joining the former lines with those of the remaining pair of corresponding lines of such triplets.*

It follows that *each of the* 10 *double-sixes of the* 3*rd kind and* 2*nd type is inherent to* 6 *different ones among the* 60 *quadrangular faces of the* 2*nd type.*

77. If a non-singular plane section $\mathfrak{S}$ of F_1, consisting of an oval and an odd branch, tends to the section $\mathfrak{C}$ of F_1 with a plane touching F_1

at a point M belonging to none of its lines, in such a way that each intermediate position is non-singular, then the oval of $\mathfrak{S}$ tends clearly to the loop of $\mathfrak{C}$, and the lines of F_1 incident to the oval or to the odd branch of $\mathfrak{S}$ are respectively inherent or non-inherent to the point M. Each of the regions of the 1st kind determined (§ 69) by the surface $\hat{F}_1$, dual of F_1, has as contour a 2-cycle of the generalized polyhedron of $\hat{F}_1$; this corresponds to a certain set of faces of the generalized polyhedron of F_1, which all have consequently the same set $\mathfrak{K}$ of lines as inherent. If conversely we consider any non-null set $\mathfrak{K}$ of lines inherent to some of the faces of F_1, we shall see that *the totality of the faces of* $\hat{F}_1$ *which correspond to the faces of* F_1 *inherent to* $\mathfrak{K}$ *constitute a single irreducible* 2-*cycle, homeomorphic to a sphere*; this is therefore the contour of a well-determined region of the 1st kind, whose points correspond to the plane sections of F_1 which contain an oval meeting all the lines of $\mathfrak{K}$, and only these.

By virtue of §§ 75, 76, $\mathfrak{K}$ can, in fact, only be a set of 16 or 12 lines. In the first case $\mathfrak{K}$ is the set of the lines skew to one, r say, of the 15 hyperbolic lines of F_1; and the faces of F_1 inherent to it are 2 pentagonal faces and 2 quadrangular faces of the 1st type, which—by adopting the notation of § 65 (Fig. 60)—can be denoted by $STS_2R''T_1$, $S^*T^*S_2^*R''^*T_1^*$, and $RS_1R'T_2$, $R^*S_1^*R'^*T_2^*$. Both the faces of $\hat{F}_1$ corresponding to the former 2 are quadrangular, since they are homeomorphic with those deduced from them by identifying the pairs of points of their sides ST, S^*T^* which correspond in the involution determined on r by F_1. Hence, if $\hat{P}$ denotes the point of $\hat{F}_1$ corresponding to a point P of F_1, we have $\hat{S} = \hat{T}$, $\hat{S}^* = \hat{T}^*$; and the faces of $\hat{F}_1$ which correspond to those 4 faces of F_1 constitute a cycle homeomorphic to a sphere, since (supposing, in accordance with § 65, $s_1 = s_1^*$, $t_1 = t_1^*$, $s_2 = t_2^*$, $t_2 = s_2^*$, and taking into account § 69) we have the following identities among their sides:

$$\hat{S}\hat{T}_1 = \hat{R}\hat{T}_2, \quad \hat{T}\hat{S}_2 = \hat{R}\hat{S}_1, \quad \hat{R}'\hat{S}_1 = \hat{R}'^*\hat{S}_1^*, \quad \hat{R}''\hat{T}_1 = \hat{R}''^*\hat{T}_1^*,$$
$$\hat{S}^*\hat{T}_1^* = \hat{R}^*\hat{T}_2^*, \quad \hat{T}^*\hat{S}_2^* = \hat{R}^*\hat{S}_1^*, \quad \hat{S}_2\hat{R}'' = \hat{T}_2^*\hat{R}'^*, \quad \hat{T}_2\hat{R}' = \hat{S}_2^*\hat{R}''^*.$$

In the second case $\mathfrak{K}$ is one of the 10 double-sixes of the 3rd kind and 2nd type, and is inherent to 6 quadrangular faces of the 2nd type of F_1 (§ 76), which can be defined as follows. The set $\mathfrak{K}$ contains a pair of complementary triplets of elliptic lines, $s_1s_2s_3$ and $t_1t_2t_3$ say; if r_{ij} is the further intersection of F_1 with the plane s_it_j, and if s_1t_1, s_2t_2, s_3t_3 are the 3 pairs of corresponding lines of our triplets, then the 6 quadrangular faces mentioned above are those determined by the sets

of lines $s_i t_j r_{lj} r_{il}$, where i, j, l is any permutation of the numbers 1, 2, 3. If the vertices of the faces are named in the manner indicated by Fig. 63, we see that the corresponding quadrangular faces of $\hat{F}_1$ constitute a 2-cycle homeomorphic to a sphere, since we have the following identities among their sides:

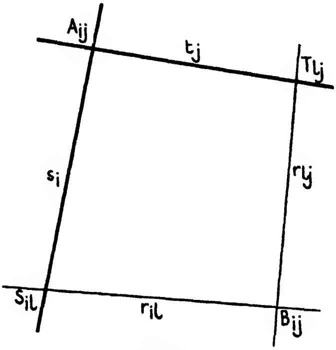


FIG. 63

$$\hat{A}_{ij}\hat{S}_{il} = \hat{A}_{il}\hat{S}_{ij}, \quad \hat{A}_{ij}\hat{T}_{lj} = \hat{A}_{lj}\hat{T}_{ij},$$
$$\hat{S}_{il}\hat{B}_{ij} = \hat{T}_{il}\hat{B}_{jl}, \quad \hat{T}_{lj}\hat{B}_{ij} = \hat{S}_{lj}\hat{B}_{li}.$$

We have still to consider the 10 triangular faces of the generalized polyhedron of F_1; each of them has a 2-cycle homeomorphic to a sphere as corresponding manifold on $\hat{F}_1$, since the latter is homeomorphic with the manifold which we deduce from the former by identifying its 3 vertices and, on each of its 3 sides, the ∞^1 pairs of points conjugate in the involution determined there by F_1.† Since the 10 triangular faces of F_1 have the null-set as inherent, the corresponding 2-cells of $\hat{F}_1$ must be the contours of 10 distinct regions of the 1st kind, each of which corresponds to a system of plane sections of F_1 containing an oval which meets none of the 27 lines; thus we obtain 10 other systems of the 1st kind, whose curves have a different behaviour in relation to the elliptic lines of F_1, in accordance with the latter part of § 74.

In order to specify the regions of the 2nd kind determined by $\hat{F}_1$, we observe that through each side e of the generalized polyhedron of $\hat{F}_1$ we have 2 sheets ϕ, ψ of this surface, on each of which e is a common side of 2 different faces of the polyhedron, say ϕ_1, ϕ_2 on ϕ and ψ_1, ψ_2 on ψ. In virtue of § 69, the side e is an edge common to the contour of 4 regions, of which 2 are of the 1st kind and 2 (not necessarily distinct) are of the 2nd kind; and, if the former have ϕ_1, ψ_1 and ϕ_2, ψ_2 as faces of their contour adjacent along e, the latter have ϕ_1, ψ_2 and ϕ_2, ψ_1 as faces of their contour adjacent along this side. From this remark and the previous results it follows that

There are 10 distinct regions of the 2nd kind determined by $\hat{F}_1$, each of

† In correspondence with these 3 sides the 2-cycle considered has therefore 3 angular segments with an extremity in common; the other 3 extremities belong to a closed cuspidal curve of the 2-cycle, which corresponds to the branch of the parabolic curve of F_1 situated in the triangular face initially considered (§§ 69, 73).

them having as contour an irreducible 2-*cycle made up of* 13 *of the* 130 *faces of the generalized polyhedron of* $\hat{F}_1$.

Let us, in fact, consider on F_1 a triangular face RST, for which we retain the notation explained in Fig. 60, denoting, moreover, by (rs), (sr), etc., the lines which are the further intersection of F_1 with the planes $r_1 s_2$, $s_1 r_2$, etc.; then we have, for instance, that the conjugate points of R_1, T', S'' in the involution determined on r_1 by F_1 are the intersections of r_1 with r_2, (rs), (rt) respectively, and that $(rs)(st)(tr)$, $(sr)(rt)(ts)$ is the pair of principal planes associated with the non-principal plane rst (§ 76). We start from the face of $\hat{F}_1$ corresponding to RST, and seek the complete contour $\mathfrak{C}$ of the region of the 2nd kind determined by $\hat{F}_1$, of which such a face is a portion. The faces of $\mathfrak{C}$ adjacent to the former along the edges which correspond to the sides of the triangular face RST are those corresponding on $\hat{F}_1$ to the three pentagonal faces of F_1 adjacent to RST. Since (cf. Fig. 60)

$$\hat{S}\hat{R}_2 = \hat{T}\hat{R}_1, \qquad \hat{T}\hat{S}_2 = \hat{R}\hat{S}_1, \qquad \hat{R}\hat{T}_2 = \hat{S}\hat{T}_1,$$

the portion of $\mathfrak{C}$ just defined has these three segments as new edges and its further contour is a hexagon, whose sides are $\hat{R}_1\hat{S}''$, $\hat{S}''\hat{T}_2$, $\hat{T}_1\hat{R}''$, $\hat{R}''\hat{S}_2$, $\hat{S}_1\hat{T}''$, $\hat{T}''\hat{R}_2$. It follows that the cells of $\mathfrak{C}$ adjacent to the last three along the sides of this hexagon are those corresponding to the quadrangular faces of the 2nd type:

$$r_1 r_2(sr)(rt), \qquad s_1 s_2(ts)(sr), \qquad t_1 t_2(rt)(ts);$$

the portion of $\mathfrak{C}$ considered till now has therefore no contour, apart from the last three edges, and the faces of $\mathfrak{C}$ adjacent to the former along these edges are those corresponding to the quadrangular faces of the 2nd type:

$$r_1 r_2(tr)(rs), \qquad s_1 s_2(rs)(st), \qquad t_1 t_2(st)(tr);$$

by adjoining to them those corresponding to the 3 quadrangular cells of the 1st type represented in Fig. 60 we obtain a 2-cycle,† so that $\mathfrak{C}$ must contain at least the 13 faces enumerated.

The result stated above is therefore completely established, on remarking that two different regions of the 2nd kind cannot have a 2-dimensional part of their contours in common (§ 69), and that—by virtue of the argument developed at the end of § 74—the number of those regions must be at least 10.

† This 2-cycle is homeomorphic with a sphere, if each of the edges belonging to it is considered as the superposition of two sides, in accordance with the previous description of the contiguity among its faces.

We have in conclusion that

The non-singular plane sections of a surface F_1 constitute the following 45 *distinct continuous systems:*

(i) 15 *systems consisting of curves with an oval, which is incident with the* 16 *lines of F_1 skew to one of its* 15 *hyperbolic lines.*

(ii) 10 *systems consisting of curves with an oval, which is incident with the* 12 *lines of one of the* 10 *double-sixes of the 3rd kind and 2nd type of F_1.*

(iii) 10 *systems consisting of curves with an oval, skew to all the* 27 *lines of F_1; two curves of this sort belong to the same or to different systems, according as the* 6 *points intersected on them by a sextuplet of elliptic lines of F_1 have or have not the same cyclic arrangement.*

(iv) 10 *systems consisting of curves without oval, which can be distinguished one from the other as in* (iii).†

The ovals of a system (iii) have as limiting positions the single points of one of the 10 regions of positive curvature of F_1, so that each of the former can be continuously deformed into one of the latter without meeting any of the 27 lines of F_1. Hence:

The ovals of all the plane curves of any one of the 10 *systems* (iii) *are completely interior to one of the* 10 *triangular faces of the generalized polyhedron of F_1.*

78. From the results of §§ 74–7 we can deduce without difficulty the classification of the surfaces F_1 in relation to their curves at infinity, i.e. from the *affine* point of view. We thus obtain in all 18 *types of cubic surfaces F_1* (having 27 distinct real lines), such that two surfaces of the same type can always be continuously deformed within the affine space one into the other, in such a way that each intermediate position is still a cubic surface of the same type; corresponding to any such surface we determine, moreover, its *group*, namely, the group of substitutions among its 27 lines induced by all the circulations of the surface which keep it of the same type.

First of all we have 4 *types of surfaces F_1 having the curve $\mathfrak{C}$ at infinity non-singular*, and respectively belonging to a continuous system of plane sections of the type (i), (ii), (iii), or (iv) (§ 77); the curve $\mathfrak{C}$ possesses an oval branch only in the first three cases.

† A less precise statement can already be found in § 13 of Klein's paper quoted in the Preface; here, however, the results are enunciated almost without proof, and the curves considered in (iv) are said to constitute a single system. Cf. also the paper by Todd quoted in § 33, where (in § 4) it is established that the oval of any plane section of F_1 can only intersect 0, 12, or 16 lines of this surface.

In the *case* (i) we have on F_1 a special hyperbolic line, r, such that each line of F_1 incident to the oval of $\mathfrak{C}$ is skew to r and conversely. The group of F_1 in the affine space is the subgroup of the transformations of Γ_1 having the line r as fixed, and is therefore (§ 58) *trirectangular*; when, in fact, we subject F_1 to a circulation transforming r into itself, a plane section of F_1 having with respect to r the same behaviour as $\mathfrak{C}$ can at the same time be deformed continuously into a plane section having the same behaviour with respect to r: and afterwards both sections can be reduced to $\mathfrak{C}$, moving on F_1 in such a way that each intermediate position has also the same behaviour with respect to r.

In the *case* (ii) we have on F_1 a special double-six of the 3rd kind and 2nd type, given by the lines meeting the oval of $\mathfrak{C}$; the corresponding group is simply isomorphic with *a symmetric group of degree* 3 (§§ 59, 60).

In the *cases* (iii) *and* (iv) all the lines of F_1 meet the odd branch of $\mathfrak{C}$, on which, in particular, the points at infinity of the 6 lines of a sextuplet of the 1st kind occur in a certain cyclic order. In both cases the group of F_1 consists of the transformations of Γ_1 which do not alter the corresponding cyclic arrangement of the 6 elliptic lines, and therefore (§ 74) it is again simply isomorphic with *a symmetric group of degree* 3.

Next we have 5 *types of surfaces* F_1 *having the curve* $\mathfrak{C}$ *at infinity singular but irreducible*, characterized by the following peculiarities of the singular point M of $\mathfrak{C}$.

(v) M is a nodal point of $\mathfrak{C}$, belonging to a pentagonal or quadrangular face of the 1st type of the generalized polyhedron of F_1. Since the $30+30 = 60$ faces of these kinds are permuted transitively by the 60 transformations of Γ_1, all the surfaces of this type are equivalent, and their group in the affine space reduces to *identity*. The same conclusion holds if

(vi) M is a nodal point of $\mathfrak{C}$, interior to one of the 60 quadrangular faces of the 2nd type of the generalized polyhedron of F_1. We obtain 3 other cases if M belongs to one of the 10 triangular faces of this polyhedron and

(vii) M is a nodal point of $\mathfrak{C}$, or

(viii) M is a cuspidal point of $\mathfrak{C}$, or

(ix) M is an isolated point of $\mathfrak{C}$. Any two of the triangular faces are equivalent with respect to Γ_1, and there are 6 transformations of Γ_1 which leave unaltered a given triangular face, constituting a subgroup simply isomorphic with a symmetric group of degree 3 (§§ 59, 76): they

transform into itself the cell formed by the elliptic points of the face, and interchange transitively both the three triangular cells and the three arcs determined by its hyperbolic and parabolic points not belonging to the contour (§ 73). It follows that in both the cases (vii) and (viii) the group of the corresponding surface contains a single non-identical transformation *of order* 2, and in the case (ix) the group is simply isomorphic with *a symmetric group of degree* 3.

The case in which $\mathfrak{C}$ *breaks up into a line r and an irreducible conic* $\mathfrak{D}$, which (§ 27) must have a real branch, gives rise to the following 6 *types*.

(x) If r is elliptic, then the conic $\mathfrak{D}$ certainly intersects r in 2 real distinct points. There are 10 transformations of Γ_1 leaving r unchanged, and they transform transitively the 5 regions determined on F_1 by the 5 tritangent planes through r, one of which contains $\mathfrak{D}$ (§ 38); it follows that in this case the group of F_1 is *of order* 2. The same conclusion can be reached (taking into account § 58) if r is hyperbolic (of the 1st kind) and intersects $\mathfrak{D}$ in 2 real points, which can be:

(xi) distinct and each belonging to a side common to two quadrangular faces of the 2nd type of the generalized polyhedron of F_1; or

(xii) distinct and each belonging to a side common to a pentagonal face and a quadrangular face of that polyhedron; or

(xiii) distinct and belonging to a side of a triangular face of that polyhedron; or

(xiv) coincident (and consequently belonging to a side of a triangular face). If, finally,

(xv) r is hyperbolic and has no real point in common with $\mathfrak{D}$, the group of F_1 in the affine space is *trirectangular*.

The case in which $\mathfrak{C}$ *breaks up into* 3 *lines*, namely, in which the plane at infinity is a tritangent plane of F_1, gives rise to 3 *types*, according as this tritangent plane is:

(xvi) A principal plane; the transformations of Γ_1 which leave this plane unaltered induce among the other 4 principal planes the 12 substitutions of even class, and the group of F_1 in the affine space is simply isomorphic with *the group of the regular tetrahedron*.

(xvii) A non-principal plane of the 1st kind, in which case (§ 37) the 3 lines may possibly have a point in common; then the group of F_1 is simply isomorphic with *a symmetric group of degree* 3 (§§ 59, 76).

(xviii) A tritangent plane of the 2nd kind; then the group of F_1 is *of order* 2, since the 30 tritangent planes of the 2nd kind are permuted transitively by the 60 transformations of Γ_1.

(ii) THE SURFACES F_2 AND $\hat{F}_2$

79. All the results of § 73 can be immediately carried over to the surfaces F_2, with the modifications implied by the fact that the generalized polyhedra of these surfaces have 6 triangular faces instead of 10 (§ 66). In particular:

The parabolic curve of a surface F_2 without real Eckardt points consists of 6 *distinct non-singular branches, inscribed in the* 6 *triangular faces of the generalized polyhedron of F_2, and enclosing six* 2*-dimensional cells constituted by the elliptic points of F_2.*

It follows (as in §§ 74, 75) that:

The 6 *triangular faces of the generalized polyhedron of F_2 have the null-set as their inherent set of lines. Every other face of such a polyhedron has a side or a vertex in common with a definite one of the triangular faces, according as it is pentagonal or quadrangular; denoting by r the hyperbolic line of F_2 which contains the common side or, respectively, the side of the triangular face opposite to the common vertex, the set of lines inherent to the pentagonal or quadrangular face initially considered is the set consisting of the* 8 *real lines of F_2 skew to r.*

Such a set is a *double-four*, and (§ 31, ii) can also be defined as the set of the real lines of one of the 9 double-sixes of the 4th kind belonging to F_2;† each of the 2 quadruplets of the double-four consists of 2 elliptic and 2 hyperbolic lines.

Moreover, by means of considerations similar to those of § 77, we see that the surface $\hat{F}_2$ dual of F_2 determines 17 regions, 15 of which are of the 1st kind while the remaining 2 are of the 2nd kind. The contour of each of the former is homeomorphic with a 2-dimensional sphere: 6 of these contours are merely the faces of the generalized polyhedron of $\hat{F}_2$ which correspond to the triangular faces of F_2, and each of the other 9 contours consists of 4 faces corresponding to the 2 pentagonal and 2 quadrangular faces of F_2 which have one of the above considered double-fours as inherent set. The contour of each of the two regions of the 2nd kind consists of 21 faces of the generalized polyhedron of $\hat{F}_2$, corresponding to those which constitute one or the other of the 2 complementary sets of 3 hexagonal cells of F_2 considered at the end of § 66. Hence:

† Thus we obtain, by the way, an intrinsic one-to-one correspondence between the 9 hyperbolic lines and the 9 double-sixes of the 4th kind belonging to F_2; the former, and therefore also the latter, can be intrinsically referred in one-to-one correspondence to the 9 tritangent planes of the 2nd kind of F_2 (§ 80).

The non-singular plane sections of a surface F_2 constitute the following 17 *distinct continuous systems:*

(i) 9 *systems consisting of curves with an oval, which is incident with the* 8 *lines of F_2 skew to one of its* 9 *hyperbolic lines.*

(ii) 6 *systems consisting of curves with an oval, which is skew to all the* 15 *real lines of F_2 and completely interior to one of the* 6 *triangular faces of the generalized polyhedron of F_2.*

(iii) 2 *systems consisting of curves with no oval, which can be distinguished one from the other in the way indicated below.*

If a non-singular plane section $\mathfrak{C}$ of F_2 has a single branch, we can orientate this branch arbitrarily; the two triplets of elliptic lines of F_2 intersect $\mathfrak{C}$ in 2 triplets of distinct points, occurring on $\mathfrak{C}$ in 2 concordant cyclic arrangements, which can be transferred to the 2 triplets of lines. The curve $\mathfrak{C}$ belongs to one or the other of the 2 systems (iii), according as it thus induces one or the other of the 2 possible associations, among the cyclic orders of such triplets of lines.

80. Any line r of F_2 belongs to 3 distinct real tritangent planes, which are all of the 2nd kind, or 2 of the 1st kind and 1 of the 2nd kind, according as r is elliptic or hyperbolic. In the second case, each transformation of Γ_2 leaving r unaltered must transform into itself the only tritangent plane of the 2nd kind through it, and therefore also each of the 2 elliptic lines of F_2 lying on it (which are one right-handed and the other left-handed); in virtue of § 42 it follows that the sub-group of Γ_2 which leaves unchanged one of the 9 hyperbolic lines (or one of the 9 tritangent planes of the 2nd kind) of F_2 is *trirectangular*.

The tritangent planes of the 1st kind of F_2 are intrinsically connected with the one-to-one correspondences among the 2 triplets of elliptic lines of F_2: any one of the former defines, in fact, one of the latter by associating 2 elliptic lines if they are in a plane through one of its 3 hyperbolic lines; and conversely. The sub-group of Γ_2 which transforms into itself one of the 6 tritangent planes of the 1st kind of F_2 is therefore simply isomorphic with a *symmetric group of degree* 3, and performs 6 distinct substitutions among the 3 lines of the plane (§ 42); hence, more generally, there is one (and only one) operation of Γ_2 which transforms any 2 given tritangent planes of the 1st kind of F_2 one into the other, inducing an arbitrary one-to-one correspondence among their lines.

Taking also into account §§ 66, 79, we easily deduce the following *classification of the cubic surfaces F_2* (with 15 real lines) *of the affine space*

into 14 *types*, as well as their *corresponding groups*, the latter having the usual meaning in relation to the former.

(i) The curve $\mathfrak{C}$ at infinity of F_2 is non-singular and has an oval incident with 8 lines of F_2: the corresponding group is *trirectangular*.

(ii) $\mathfrak{C}$ is non-singular and has an oval skew to all the lines of F_2: the corresponding group is simply isomorphic with a *symmetric group of degree* 3.

(iii) $\mathfrak{C}$ is non-singular and with a single branch: the corresponding group is *of order* 18, and its transformations are characterized by the property of performing any two substitutions *of the same class* upon the two complementary triplets of elliptic lines of F_2.

(iv) $\mathfrak{C}$ has a nodal point interior to a pentagonal or quadrangular face of the generalized polyhedron of F_2: the corresponding group reduces to *identity*.

(v) $\mathfrak{C}$ has a nodal point interior to a triangular face of the generalized polyhedron of F_2: the corresponding group is *of order* 2.

(vi) $\mathfrak{C}$ has a cuspidal point: the corresponding group is still *of order* 2.

(vii) $\mathfrak{C}$ has an isolated double point: the corresponding group is isomorphic with a *symmetric group of degree* 3.

(viii) $\mathfrak{C}$ breaks up into an elliptic line of F_2 and an irreducible conic, which necessarily intersects this line in 2 distinct real points: the corresponding group is *trirectangular*.

$\mathfrak{C}$ breaks up into a hyperbolic line of F_2 and an irreducible conic, which necessarily has a real branch, and determines on the line 2 points which are

(ix) distinct, real, and not belonging to the contour of any triangular face of the generalized polyhedron of F_2: the corresponding group is *of order* 2;

(x) distinct and belonging to the contour of a triangular face: the corresponding group is *of order* 2;

(xi) coincident (in a parabolic point of the hyperbolic line): the corresponding group is still *of order* 2;

(xii) conjugate complex: the corresponding group is *trirectangular*.

(xiii) $\mathfrak{C}$ breaks up into 3 hyperbolic lines of F_2, possibly belonging to a pencil: the corresponding group is simply isomorphic with a *symmetric group of degree* 3.

(xiv) $\mathfrak{C}$ breaks up into 1 hyperbolic and 2 elliptic lines: the corresponding group is *trirectangular*.

(iii) The surfaces F_3 and $\hat{F}_3$

81. Let us consider a surface F_3 without real Eckardt points, for which we keep the notation of § 67; its two tritangent planes δ, ϵ are

of the 3rd kind: and we denote by D, E respectively their principal points (§ 31), neither of which—with our assumption—can lie on r, and by λ, μ the planes touching F_3 at the parabolic points L, M of r. We can, for instance, suppose that the 7 planes $\epsilon\lambda\alpha\beta\gamma\mu\delta$ occur in their pencil in this cyclic arrangement, so that L and M respectively belong to the sides A_1A_2 and C_1C_2 of the generalized polyhedron of F_3.

This polyhedron has two triangular faces of vertices A_1AA_2, of which one—containing the point E—also contains completely the conic intersected by λ on F_3 outside r (§ 67); it follows, in view of § 71, that the parabolic curve of F_3 goes simply through L touching r at this point, and in the neighbourhood of L it lies completely within the triangular face A_1AA_2 which does not contain E.

By a reasoning similar to that indicated in § 73 for the surfaces F_1, we obtain the result that:

The parabolic curve of a surface F_3 without real Eckardt points, consists of 4 distinct non-singular branches; each of these belongs to one of the 4 triangular faces of the generalized polyhedron of F_3, and touches its 2 hyperbolic sides of the 1st kind: two of them (those belonging to a triangular face which does not contain one of the points D, E considered above) are, moreover, tangent to the remaining hyperbolic side of the 2nd kind. On F_3 there are 4 distinct simply connected regions of elliptic points, and a single connected region of hyperbolic points having as contour the 4 branches of the parabolic curve.

The hyperbolic points of a triangular face form 2 or 3 connected regions, according as this face does or does not contain one of the points D, E; and it is possible to pass continuously from one case to the other by means of a deformation of F_3.† One of the regions, which we call of the 1*st kind*, has on its contour the point common to the 2 hyperbolic sides of the 1st kind of the triangular face; the other one (or two), which we call of the 2*nd kind*, will have a segment of the contour belonging to the hyperbolic line of the 2nd kind of F_3. We see, moreover (as in §§ 74, 75), that:

The 4 triangular faces of the generalized polyhedron of F_3 have the null-set as their inherent set of lines. Each of the other 4 hexagonal faces of such a polyhedron has an inherent set of 4 lines, given by the two elliptic

† In fact, by suitably deforming F_3, it is possible to let one of the points D, E cross the hyperbolic line of the 2nd kind; then such a point becomes an Eckardt point of the corresponding position of F_3, the parabolic curve of this surface acquires a nodal point in it (§ 71), and μ or λ comes into coincidence with δ or ϵ.

lines of F_3 and by the 2 hyperbolic lines of the 1st kind which have no points in common with the contour of that face.

Finally, by means of considerations of the same kind as those developed in § 77, we can deduce that the surface $\hat{F}_3$ dual to F_3 determines 7 regions, of which only one is of the 2nd kind, and has the whole $\hat{F}_3$ as its contour. Four of the remaining regions have each as contour a single face of the generalized polyhedron of $\hat{F}_3$, corresponding to one of the 4 triangular faces of F_3, and the contours of the other 2 each consist of 2 faces of $\hat{F}_3$, corresponding to 2 hexagonal faces of F_3 with the same 6 vertices. Hence:

The non-singular plane sections of a surface F_3 constitute the following 7 distinct continuous systems:

(i) *Two systems consisting of curves with an oval, which is incident with the 2 elliptic lines of F_3 and with the 2 hyperbolic lines of the 1st kind belonging to one of the 2 tritangent planes of the 1st kind of F_3.*

(ii) *Four systems consisting of curves with an oval, which is skew to all the 7 real lines of F_3 and completely interior to one of the 4 triangular faces of the generalized polyhedron of F_3.*

(iii) *A single system of curves without oval.*

82. *In the affine space the cubic surfaces F_3* (with 7 real lines) *can be classified into* 23 *types*, depending on the following possibilities for their curve $\mathfrak{C}$ at infinity.

(i) $\mathfrak{C}$ is non-singular, and has an oval incident with 4 lines of F_3.

(ii) $\mathfrak{C}$ is non-singular, and has an oval skew to all the lines of F_3.

(iii) $\mathfrak{C}$ is non-singular and with a single branch.

(iv) $\mathfrak{C}$ has a nodal point interior to a hexagonal face of the generalized polyhedron of F_3.

(v) $\mathfrak{C}$ has a nodal point interior to a region of the 1st kind of a triangular face of the generalized polyhedron of F_3.

(vi) $\mathfrak{C}$ has a cuspidal point on the contour of such a region.

(vii) $\mathfrak{C}$ has a nodal point interior to a region of the 2nd kind of a triangular face of the generalized polyhedron of F_3.

(viii) $\mathfrak{C}$ has a cuspidal point on the contour of such a region.

(ix) $\mathfrak{C}$ has an isolated double point.

(x) $\mathfrak{C}$ breaks up into an elliptic line of F_3 and an irreducible conic, which necessarily intersects this line in two distinct real points.

$\mathfrak{C}$ may break up into a hyperbolic line h of the 1st kind of F_3 and an irreducible conic $\mathfrak{D}$, which necessarily has a real branch. The two

points H_1, H_2, intersections of h and $\mathfrak{D}$, if real and distinct, determine on $\mathfrak{D}$ an arc $H_1 H_2$ containing the intersections K_1, K_2 of $\mathfrak{D}$ with the left- and right-handed elliptic lines of F_3 respectively: and we can suppose H_1, H_2 to have been so named that the points $H_1\ K_1\ K_2\ H_2$ occur on $\mathfrak{D}$ in this order; then, if I_1, I_2 are the respective intersections of h with the hyperbolic lines of the 1st and 2nd kind of F_3 incident to it, we have to distinguish two cases—(xi) and (xii)—according as the points $I_1\ H_1\ H_2\ I_2$ occur on h in this order or in the order $I_1\ H_2\ H_1\ I_2$. A similar distinction can be made even if H_1, H_2 coincide, leading to two other cases, which we denote by (xiii) and (xiv); and a (xv)th case arises if H_1, H_2 are conjugate complex.

If $\mathfrak{C}$ breaks up into the hyperbolic line r of the 2nd kind and an irreducible conic $\mathfrak{D}$ intersecting r in R_1, R_2, we have the following cases:

(xvi) R_1, R_2 are distinct, real, and do not belong to the contour of any triangular face of the generalized polyhedron of F_3;

(xvii) R_1, R_2 are distinct and belong to the common side of two adjacent triangular faces;

(xviii) R_1, R_2 coincide (in a parabolic point of r).

(xix) R_1, R_2 are conjugate complex, but $\mathfrak{D}$ has a real branch.

(xx) $\mathfrak{D}$ has not a real branch.

$\mathfrak{C}$ may finally break up into 3 lines, the plane at infinity being:

(xxi) a tritangent plane of the 1st kind;

(xxii) a tritangent plane of the 2nd kind;

(xxiii) a tritangent plane of the 3rd kind.†

The *group* inherent to each of the 23 types of cubic surfaces F_3 reduces to the *identical transformation*, except the types (i), (x), (xvi), (xvii), (xviii), (xix), (xxi), (xxiii), for each of which it is *of order* 2, and the types (iii), (xx), (xxii), for each of which it is *trirectangular*.

(iv) THE SURFACES F_4, F_5 AND $\hat{F}_4$, $\hat{F}_5$

83. Let us consider a surface F_4 or F_5 without real Eckardt points, and one, r say, of its 3 real lines. Through r we take the parabolic planes λ and μ (intersecting the surface further along two conics $\mathfrak{L}$ and $\mathfrak{M}$, which touch r at its parabolic points L and M respectively), the tritangent plane of the 1st kind α (intersecting the surface further along its remaining 2 real lines a_1, a_2, which determine on r two real points A_1, A_2 harmonically separating L, M), and, finally, the two tritangent planes of the 3rd kind of F_4 or the two tritangent planes of the 3rd kind

† In this case only one of the 3 lines at infinity is real; in both the cases (xxi) and (xxiii) the 3 lines may possibly have a point in common.

and 1st type of F_5 (§ 52), β and γ say, whose principal points we call B and C respectively; and we can suppose these planes to occur in their pencil in the cyclic arrangement $\alpha\lambda\beta\gamma\mu$ (§§ 48, 52).

If a plane δ describes the oriented angle $\alpha\lambda\beta$, its intersection with F_4 or F_5 residual to r is a real conic $\mathfrak{D}$, starting from the pair of lines $a_1 a_2$ and reducing (in the real field) to the real point B, every intermediate position of $\mathfrak{D}$ being non-singular. When δ is between α and λ, $\mathfrak{D}$ intersects r in two distinct points of the segment $A_1 L A_2$, which determine on it two arcs. One of these reduces to the point L when $\delta \to \lambda$ and $\mathfrak{D} \to \mathfrak{L}$, and describes one of the two faces of the generalized polyhedron of our surface having the segment $A_1 L A_2$ as a side; the locus of the other arc and of the conics $\mathfrak{D}$ arising from the planes δ comprised between λ and β is the other one of the two faces, which therefore contains the whole conic $\mathfrak{L}$ and the point B. It follows, by virtue of § 71, that the former face has an arc of parabolic points, touching r at L; moreover, the second face must also have an arc of parabolic points, separating the elliptic point B from the hyperbolic points of the contour.

Since a similar argument can be applied for the sections of the surface with the planes of the angle $\alpha\mu\gamma$, and also if we substitute a_1 or a_2 for the line r, we have, taking also into account §§ 6, 50, 54, and 71, that:

The parabolic curve of a surface F_4 or F_5 without real Eckardt points consists of 4 distinct non-singular branches; each of the 4 triangular faces of the generalized polyhedron contains one such branch,† *which touches a certain number i of its sides* ($0 \leqslant i \leqslant 3$), *and encloses a region of positive curvature in which there are exactly $3-i$ points whose tangent plane passes through a real line of the cubic surface. Both these points and the points of contact of the 3 real lines with the branches of the parabolic curve are 6 in number, and are the vertices of 2 plane quadrilaterals.*

Figs. 46 and 45 show that, if a surface F_4 or F_5 is sufficiently near to one real and two conjugate complex planes, the number i *has the 4 distinct values* 0, 1, 2, 3 in correspondence with the 4 triangular faces of its generalized polyhedron; these values can, however, alter if we subject the surface to a continuous variation:‡ and we omit to enumerate all the cases that can thus arise. We can instead observe the following consequences of the above developments:

† In § 10 of Zeuthen's paper quoted in the Preface it is stated, without proof, that the three lines of a surface F_4 or F_5 determine on it *three* triangles containing the branches of the parabolic curve.

‡ Cf. the similar remark made in § 81 for the surfaces F_3.

A tritangent plane of the 3rd kind of a cubic surface F_5 is of the 1st or 2nd type, according as its principal point belongs to the non-ovoidal or to the ovoidal piece of F_5.

By means of considerations of the same kind as those developed in § 77, we can deduce that the surface $\hat{F}_4$ or $\hat{F}_5$, dual of F_4 or F_5, determines 5 or 6 regions respectively, of which only one is of the 2nd kind, and has the whole surface $\hat{F}_4$ or $\hat{F}_5$ as its contour. Four other regions have each as contour a single face of the generalized polyhedron of $\hat{F}_4$ or $\hat{F}_5$, corresponding to one of the 4 triangular faces of F_4 or F_5; the remaining region which arises for $\hat{F}_5$ has as contour the portion of this surface corresponding to the ovoidal piece of F_5. Hence:

The non-singular plane sections of a surface F_4 or F_5 respectively form 5 or 6 distinct continuous systems, one of which is generated by the sections with no oval, while the others consist of curves with an oval, which is skew to each of the 3 real lines of the surface and either completely interior to one of the 4 triangular faces of its polyhedron, or traced upon the ovoidal piece of F_5.

From here, taking also into account §§ 49, 54, we could easily derive the *classification of the surfaces F_4, F_5 in the affine space*, and the determination of the *groups* inherent to the separate types. We leave this to the reader.

IV

THE SYLVESTER PENTAHEDRON AND THE CUBIC SURFACES WITH HOMOGRAPHIC TRANSFORMATIONS INTO THEMSELVES

XIII. Sylvester's canonical form and exceptional cubic surfaces

84. We shall prove the following theorem due to Sylvester:†

The equation of the general cubic surface F can be reduced in one and only one way to the canonical form

$$\sum_{i=0}^{4} \lambda_i x_i^3 = 0, \tag{1}$$

where $x_i = 0$ *are the equations of 5 planes* π_i *independent 4 by 4, and the* λ_i*'s are 5 non-zero constants.*

The equation (1), in fact, represents a cubic surface dependent—like the general cubic surface of [3]—on 19 non-homogeneous parameters; we shall see that the equation of the former surface can be reduced *in a unique way* to the form (1), whence the above theorem follows at once, by virtue of the well-known Plücker-Clebsch criterion.‡ We remark for this purpose that, by absorbing convenient factors into the left-hand sides of the equations of the 5 planes π_i, we can suppose the linear relation between them reduced to the form

$$\sum_{i=0}^{4} x_i = 0. \tag{2}$$

Hence, as is shown by a simple calculation, the equation of the Hessian surface of (1) is

$$\sum_{i=0}^{4} \frac{1}{\lambda_i x_i} = 0.$$

This surface passes simply through the 10 edges of the pentahedron determined by the planes π_i; it has the 10 vertices of such a pentahedron as ordinary double points, and no other singular point if (as we can suppose)

$$\sum_{i=0}^{4} \frac{1}{\pm\sqrt{\lambda_i}} \neq 0,$$

† For the history of this theorem cf. Meyer's article quoted in the Preface, § 3.

‡ On which cf., for instance, F. Severi, 'Sulla compatibilità dei sistemi di equazioni algebriche ed analitiche', *Rendic. R. Acc. Lincei*, (VI) vol. 17 (1933), pp. 3–10. We can also say simply that the above property shows that the cubic surfaces (1) *and their limits* constitute an irreducible algebraic system *of dimension 19*, belonging to—and therefore *coinciding* with—the ∞^{19} linear system of all cubic surfaces.

i.e. (§ 109) if the surface (1) is non-singular:† which implies that the pentahedron is uniquely determined by the surface. This, on the other hand, cannot have another similar representation

$$\sum_{i=0}^{4} \lambda_i' x_i^3 = 0$$

leading to the same pentahedron, but with the constants λ_i' non-proportional to the λ_i's, since otherwise we should derive from the two representations an identity

$$\sum_{i=0}^{4} \mu_i x_i^3 = 0$$

(with not all the μ_i's equal to zero) which must be a direct consequence of (2), and this is obviously absurd.

In the following paragraphs we shall complete the theorem just proved by determining the cubic surfaces which are exceptional in connexion with it, that is, the non-singular cubic surfaces for which the Sylvester representation is either indeterminate or impossible.

85. Let us consider, if it exists, a non-singular cubic surface F, whose equation is obtainable in more than one way (and, consequently, in an infinity of ways) by equating to zero the sum of the cubes of 5 linear forms. The 5 linear forms appearing in any one of such representations of F cannot be 4 by 4 independent (§ 84), so that they must identically satisfy one—and, on account of the non-singularity of F, only one—linear relation involving 4 or less of them. It follows that *F belongs to the pencil determined by a plane π counted three times and a cubic cone K having the vertex V outside π.* We shall now give some intrinsic characterizations of the cubic surfaces of this type, which we shall describe as *cyclic*, calling V their *centre* and π their *fundamental plane*, for the following reason.

The only irreducible cubic surfaces, which are transformed into themselves by a cyclic homology of period 3 (*and by the inverse of this*) *are the cyclic cubic surfaces.*

In fact, by choosing in [3] convenient projective homogeneous coordinates $(x_0\ x_1\ x_2\ x_3)$, we can reduce the equations of any given cyclic homology of period 3 to the form

$$x_0' : x_1' : x_2' : x_3' = \epsilon x_0 : x_1 : x_2 : x_3,$$

† If (1) is singular, the only further singular points of the Hessian are the singular points of the surface (1) itself.

where ϵ is a primitive cube root of unity. A cubic surface having the equation

$$\phi_0 x_0^3+\phi_1 x_0^2+\phi_2 x_0+\phi_3 = 0, \tag{3}$$

where ϕ_i is a form of degree i in x_1, x_2, x_3, is transformed by the above homology into the surface of equation

$$\phi_0 x_0^3+\epsilon\phi_1 x_0^2+\epsilon^2\phi_2 x_0+\phi_3 = 0;$$

ϕ_3 does not vanish identically, as the former is irreducible: hence this surface coincides with the latter if, and only if, ϕ_1 and ϕ_2 vanish identically, i.e. if the cubic surface (3) is cyclic, in which case the centre and fundamental plane of the homology are respectively its centre and fundamental plane.

86. The cyclic cubic surfaces which are not cones can also be characterized as the non-ruled cubic surfaces F for which *there is a point V, not lying on F, such that every tangent of F through it is a principal tangent of F*; the point V is then the centre of F, and the locus $\mathfrak{L}$ of the points of contact of the tangents through V is a plane cubic, which is the section of F with its fundamental plane.

The generic point M of any component of $\mathfrak{L}$ is, indeed, a simple point of F, in which this surface has 3 intersections with MV (and no more than 3, since V does not belong to F); it follows that both the polar quadric and the polar plane of V with respect to F go through M, the former intersecting F along $\mathfrak{L}$ counted twice (and no more than that, since the line MV absorbs only 2 of the tangents through V of the section of F with a generic plane through it). $\mathfrak{L}$ is therefore a plane cubic without multiple components, and the polar quadric reduces to the plane π of $\mathfrak{L}$ counted twice. Using projective homogeneous coordinates $(x_0\, x_1\, x_2\, x_3)$ such that $x_0 = 0$ is the equation of π and $(1\ 0\ 0\ 0)$ are the coordinates of the point V (which certainly does not lie on π), and representing F again by the equation (3), the polar quadric of V has the equation $3\phi_0 x_0^2+2\phi_1 x_0+\phi_2 = 0$; hence ϕ_1 and ϕ_2 must vanish identically, and F is cyclic, having V as centre and π as fundamental plane.

It is obvious that, conversely, every cyclic cubic surface satisfies the condition stated above.

87. A last characterization of the cyclic cubic surfaces is given by the following theorem:

The only non-singular cubic surfaces for which there is a point V such that all their sections with the planes through V have the same modulus

are the cyclic cubic surfaces possessing V as centre; conversely, all the sections of a cyclic cubic surface (not a cone) with the planes through the centre are equianharmonic.

The second part of this theorem is an immediate consequence of a well-known property,† since each of the sections under consideration is transformed into itself by a cyclic homology of period 3.‡

In order to establish the other part we consider a non-singular cubic surface F, for which the condition enunciated is satisfied. Then the section of F with a generic plane through V, being non-singular, has a modulus $\neq 1$, so that the ∞^1 singular sections of F with the tangent planes through V have an indeterminate modulus, namely, each of them must possess a cuspidal point or a higher singularity (but not in general a triple point, since F is non-ruled). In other words, the locus $\mathfrak{L}$ of the points of contact of the tangents of F through V belongs to the parabolic curve of F. In the generic point M of each component of $\mathfrak{L}$, the tangent of $\mathfrak{L}$ and the line MV are conjugate tangents of F; hence one of them must coincide with the line touching at M the section $\mathfrak{C}$ of F with the tangent plane at this point. If, for every position of M upon an irreducible component of $\mathfrak{L}$, $\mathfrak{C}$ had a tacnode at M, the curve $\mathfrak{C}$ itself would contain a line, which is incompatible with the non-singularity of F. In virtue of § 70 it is therefore impossible for $\mathfrak{C}$ and $\mathfrak{L}$ to have always the same tangent at M; consequently MV always coincides with the line touching $\mathfrak{C}$ at M, so that it is a principal tangent of F. On the other hand, the point V cannot lie on F, since otherwise the section of F with the polar quadric of V would consist of lines through F, and F would have a singular point at V. Then, by applying § 86, we deduce at once the first part of our theorem.

88. It is easy to prove that, with the notation of § 85,

A cyclic cubic surface F which is not a cone has a single projective invariant, given by the modulus of the cone K associated to it. The Hessian surface of F consists of the fundamental plane π and of the cubic cone Hessian of K; if F is non-singular (which is equivalent to saying that K has no multiple generator), such a Hessian cone is reducible when, and only when, K is equianharmonic, in which case we call F also equianharmonic.

† Cf., for instance, F. Enriques—O. Chisini, op. cit. in § 62, vol. ii (Bologna, Zanichelli, 1918), p. 234.

‡ By means of the plane representation of a cyclic cubic surface we see, consequently, that on a plane there are *nets of equianharmonic cubic curves*.

An equianharmonic cubic surface has the *canonical equation*

$$x_0^3+x_1^3+x_2^3+x_3^3 = 0, \tag{4}$$

and is cyclic in 4 different ways: its 4 centres and 4 fundamental planes are the vertices and faces of a single tetrahedron, constituting its Hessian surface.

A cyclic non-singular and non-equianharmonic cubic surface has the *canonical equation*

$$x_0^3+(x_1^3+x_2^3+x_3^3-3\lambda x_1 x_2 x_3) = 0 \quad [\text{with } \lambda(\lambda^3+8)(\lambda^3-1) \neq 0], \tag{5}$$

and possesses a single centre and a single fundamental plane, which are immediately defined by its Hessian surface.†

Taking also into account § 85, and noticing that (§ 96) the equation of an irreducible plane cubic curve can be expressed in ∞^2 different ways by equating to zero the sum of the cubes of 4 linear forms, we have that

The only non-singular cubic surfaces for which the Sylvester representation is indeterminate are the cyclic ones. Conversely, a non-singular cyclic cubic surface F has ∞^2 Sylvester pentahedra, which all have the fundamental plane of F as one of the 5 faces and the other 4 faces intersecting in the centre of F; there is only an obvious alteration if F is equianharmonic, in which case its Sylvester pentahedra constitute 4 distinct ∞^2 systems (in conformity with the fact that F has then 4 centres and 4 fundamental planes), and a further ∞^3 system whose pentahedra consist of the 4 fundamental planes together with another arbitrary plane.

89. A non-singular cubic surface F which belongs to the ∞^4 linear system Σ determined by 5 planes π_i ($i = 0, 1,..., 4$) independent 4 by 4, each counted 3 times, but not to a system determined by 4 of these 5 planes, will be called *generic*; in such a case, both Σ and the 5 planes π_i are uniquely defined by F (§ 84).

If *G is any non-singular cubic surface of* [3], we have by virtue of § 84 that—in every neighbourhood of G within the totality of the cubic surfaces of [3]—there is some cubic surface F, distinct from G, which is generic; in other words, *G can be considered as a limiting position of a variable suitably chosen 'generic' cubic surface F.* It is obviously possible to restrict conveniently the variability of F in the neighbourhood of G, in such a way that, when F tends to G, both Σ and each of the

† The equations (4), (5) follow immediately from the well-known canonical equation of the (equianharmonic or general) plane cubic curve; on this subject cf., for instance, F. Enriques—O. Chisini, op. cit. in § 62, vol. ii, p. 217.

planes π_i inherent to F, as well as each of the lines $r_{ij} = \pi_i \pi_j$ and each of the points $R_{ijl} = \pi_i \pi_j \pi_l$, have defined limiting positions, say Θ and χ_i, s_{ij}, S_{ijl} respectively. Then G and χ_i^3 belong to Θ, s_{ij} belongs to χ_i, χ_j, and S_{ijl} belongs to χ_i, χ_j, χ_l, s_{jl}, s_{li}, s_{ij}; moreover, since, for instance, R_{012} and each point of r_{34} are conjugate with respect to every polar quadric of F, it follows that S_{012} and each point of s_{34} are conjugate with respect to every polar quadric of G: so that S_{012} *cannot belong to* s_{34}, as S_{012} cannot be a point common to all these polar quadrics, G being non-singular. It follows at once that, for instance, s_{01} *and* s_{34} *cannot coincide*. We shall now prove that:

If the 5 planes χ_i are distinct, the 5 cubic surfaces χ_i^3 must be linearly independent.

In fact, on our hypothesis, the 5 planes χ_i cannot belong to a pencil, since otherwise s_{34} would coincide with the axis of this pencil, to which S_{012} would belong. If the 5 planes χ_i belong to a net and, for instance, χ_0, χ_1, χ_2 are independent, then the only surfaces of the net determined by χ_0^3, χ_1^3, χ_2^3 which have a triple line are those of the pencils determined by two of χ_0^3, χ_1^3, χ_2^3, and none of them can belong to the pencil determined by χ_3^3 and χ_4^3. Finally our theorem is algebraically evident if 4 of the 5 planes χ_i are linearly independent.

We remark that, if the 5 planes χ_i are distinct, the system Θ is therefore that determined by the 5 cubic surfaces χ_i^3; so that the 5 planes χ_i cannot belong to a net, since Θ contains the non-singular surface G. With that assumption *the surface G is generic or cyclic*, according as the planes χ_i are or are not 4 by 4 linearly independent, and consequently it has respectively just one or (at least) ∞^2 Sylvester representations; in other words, the above limiting process applied to any non-singular surface G for which the Sylvester representation is impossible must lead to 5 planes χ_i which are not all distinct.

90. In order to study the possibility of surfaces G of this type, we begin by establishing the following proposition (which could be extended, on account of its differential character):

On a rational curve $\mathfrak{C}$, m distinct points $P_1, P_2,..., P_m$—if counted each n times $(2 \leqslant m \leqslant n)$—constitute m linearly independent sets of n points $P_1^n, P_2^n,..., P_m^n$, which therefore determine a g_n^{m-1}. If the points $P_1, P_2,..., P_m$ move on $\mathfrak{C}$, remaining distinct, and tend in an arbitrary manner to one and the same point P, then the above considered g_n^{m-1} has a well-defined limit, given by the totality of the sets of n points of $\mathfrak{C}$ having P as a point of multiplicity (at least) $n-m+1$.

By means of a convenient birational transformation, we can always reduce $\mathfrak{C}$ to a rational normal curve of $[n]$. Then the points $P_1, P_2,\ldots, P_m$ are linearly independent, namely, they belong to an $[m-1]$; the polar $[n-m]$ of this in the polarity of $[n]$ which transforms each point of $\mathfrak{C}$ into its osculating prime, is manifestly the axis of the ∞^{m-1} linear system of primes intersecting g_n^{m-1} on $\mathfrak{C}$. When $P_1, P_2,\ldots, P_m$ tend to P, the $[m-1]$ tends to the space of dimension $m-1$ osculating $\mathfrak{C}$ at P:† hence $[n-m]$ tends to the polar space of the latter, namely, to the space of dimension $n-m$ osculating $\mathfrak{C}$ at P; and g_n^{m-1} tends to the linear series intersected on $\mathfrak{C}$ by the primes through this space, that is, to the totality of the sets of n points of $\mathfrak{C}$ having P as a point of multiplicity at least $n-m+1$.

91. With the notation of § 89 we have that:

If χ_0 and χ_1 coincide, the pencil determined by π_0^3 and π_1^3 tends to the pencil Φ whose surfaces are composed of χ_0 counted twice and a plane describing the pencil of axis s_{01}.

In fact the surfaces of the former pencil intersect a g_3^1 on a line generically chosen in [3], of which § 90 gives the limit immediately; hence we see that χ_0 must be at least a double component of every limiting surface, which, on the other hand, must have s_{01} as a triple line, since r_{01} is a triple line of every surface of the pencil.

If among the 5 planes χ_i only χ_0 and χ_1 coincide, the net Ψ determined by χ_2^3, χ_3^3, χ_4^3 can contain no surface of the pencil Φ.

In fact the planes χ_2, χ_3, χ_4 have in common the point S_{234}, which is therefore triple for all the surfaces of the net Ψ and cannot belong to the line s_{01} (§ 89); the only surface of the pencil Φ having a triple point outside s_{01} is χ_1^3: but this surface certainly does not belong to the net Ψ, since χ_1, χ_2, χ_3, χ_4 are 4 distinct planes (which *a priori* can be either independent or not).

The ∞^4 linear system Θ is therefore the join of Φ and Ψ, and the 4 planes χ_1, χ_2, χ_3, χ_4 are independent since Θ contains the non-singular surface G; assuming these planes as fundamental planes for the coordinates (x_0, x_1, x_2, x_3), the equation of G can be reduced to the form

$$(x_1^3+x_2^3+x_3^3)-x_0^2(\lambda_0 x_0+3\lambda_1 x_1+3\lambda_2 x_2+3\lambda_3 x_3) = 0, \tag{6}$$

† If, in fact, $[m-1]_0$ is any space of dimension $m-1$ belonging to the *general limit* of $[m-1]$, all the primes through it must have at least m intersections with $\mathfrak{C}$ at the point P, so that $[m-1]_0$ must coincide with the space of dimension $m-1$ osculating $\mathfrak{C}$ at P, and is therefore the *regular limit* of $[m-1]$ (for the meaning of the expressions in italics, cf. the author's paper quoted in § 2).

and the coefficients λ_i satisfy the condition

$$\lambda_0+2(\lambda_1^{\frac{3}{2}}+\lambda_2^{\frac{3}{2}}+\lambda_3^{\frac{3}{2}}) \neq 0, \tag{7}$$

expressing the non-singularity of G. The Hessian H of (6) is given by the equation

$$x_1 x_2 x_3 \sum_{i=0}^{3} \lambda_i x_i + x_0^2(\lambda_1^2 x_2 x_3+\lambda_2^2 x_3 x_1+\lambda_3^2 x_1 x_2) = 0; \tag{8}$$

and a simple calculation shows that, on account of (7), it has no singular points outside the faces χ_1, χ_2, χ_3, χ_4 of the fundamental tetrahedron. In order to discuss the singularities of H we distinguish different possibilities, according as none, one, two, or all the λ_1, λ_2, λ_3 are equal to zero.

(i) If $\lambda_1 \lambda_2 \lambda_3 \neq 0$, H is irreducible and has the 7 double points

$$A_0(1,0,0,0), \qquad A_1(0,1,0,0), \qquad A_2(0,0,1,0), \qquad A_3(0,0,0,1),$$
$$B_1(0,0,\lambda_3,-\lambda_2), \qquad B_2(0,-\lambda_3,0,\lambda_1), \qquad B_3(0,\lambda_2,-\lambda_1,0)$$

(the last 6 of which are the vertices of a plane quadrilateral), and no other singular point. The points A_1, A_2, A_3 are double biplanar for H, and their edges—which belong to H—are concurrent in A_0; besides these 3 lines, H contains the 4 sides of the quadrilateral considered above.

(ii) If $\lambda_3 = 0$, but $\lambda_1 \lambda_2 \neq 0$, H breaks up into the plane $A_0 A_1 A_2$ $(x_3 = 0)$ and a cubic cone (without multiple generators) of vertex A_3. Equation (6) shows that G is then a cyclic non-equianharmonic cubic surface.

(iii) If all the λ_1, λ_2, λ_3, or even only two of them, are equal to zero, H consists of 4 independent planes, and—in virtue of (6), (7)—G is an equianharmonic cubic surface.

(i) is therefore the only new case; it gives rise to two distinct possibilities, according as $\lambda_0 \neq 0$ or $\lambda_0 = 0$, i.e. according as G does not or does contain A_0. If $\lambda_0 \neq 0$, we can let λ_0 assume the value -1 by altering x_0 by a convenient factor, and we obtain for G the *canonical equation*

$$x_0^3+x_1^3+x_2^3+x_3^3-3x_0^2(\lambda_1 x_1+\lambda_2 x_2+\lambda_3 x_3) = 0 \quad (\lambda_1 \lambda_2 \lambda_3 \neq 0), \tag{6_1}$$

showing that the surfaces under consideration depend on 18 parameters. If $\lambda_0 = 0$, the *canonical equation* of G becomes

$$x_1^3+x_2^3+x_3^3-3x_0^2(\lambda_1 x_1+\lambda_2 x_2+\lambda_3 x_3) = 0 \quad (\lambda_1 \lambda_2 \lambda_3 \neq 0), \tag{6_2}$$

in which λ_1, λ_2, λ_3 can still be altered by an arbitrary (non-zero) factor; the surfaces of this type depend, therefore, on 17 parameters.

The two types can be geometrically distinguished by noticing that, in virtue of (7), (8), H has in A_0 a tangent quadric cone which touches H along A_0A_1, A_0A_2, A_0A_3 and intersects it further in a conic, which is irreducible in the first case and composed of two distinct lines in the second case. In both cases the non-singular cubic surface G is neither generic nor cyclic, so that

The surface G represented by (6) *has no Sylvester representation; when any generic cubic surface F tends to G, the Sylvester pentahedron of the former tends to the fundamental tetrahedron of the coordinates, whose face $x_0 = 0$ has to be counted twice; such a tetrahedron is connected with the Hessian surface H of G in the manner explained above.*†

92. We can exclude the case $\chi_0 = \chi_1$, $\chi_2 = \chi_3$, with no other coincidence occurring among the 5 planes χ_i. In fact, since s_{01} and s_{23} cannot coincide (§ 89), the surface χ_4^3 and the limits of the pencils determined by π_0^3, π_1^3 and by π_2^3, π_3^3 (§ 91) are independent, as is easily seen by considering the sections with a fixed generic plane, which also are independent; but then G, which belongs to the ∞^4 linear system Θ determined by χ_4^3 and these limiting pencils, would have any point common to χ_0, χ_2, and χ_4 as a singular point.

93. *If χ_0, χ_1, χ_2 coincide in a single plane, the cubic surfaces which are the limits of those surfaces of the net N determined by π_0^3, π_1^3, π_2^3 are all composed of the plane χ_0 counted (at least) once and a quadric cone, of vertex S_{012}, touching χ_0 along the line s_{01}.*

Each of the limiting surfaces must, in fact, contain χ_0 as a component, in virtue of § 90, and must have S_{012} as a triple point, since the point R_{012} is triple for all the surfaces of N. It is obvious that each point of r_{01} and each point of π_2 are conjugate with respect to all the polar quadrics of every surface of N; hence each point of s_{01} and each point of χ_2 ($= \chi_0 = \chi_1$) are conjugate with respect to all the polar quadrics of every limiting cubic surface, i.e. these polar quadrics are cones touching χ_0 along s_{01}, and the limiting surfaces are of the type stated above. Since these surfaces must have the same behaviour with

† A first attempt to consider the limit of a generic cubic surface when two faces of its pentahedron tend to coincide was made incidentally by F. E. Eckardt, 'Ueber diejenigen Flächen dritten Grades, auf denen sich drei gerade Linien in einem Punkte schneiden', *Math. Ann.*, vol. 10 (1876), pp. 227–72 (p. 234); here, however, owing to the special limiting process employed, only surfaces with a uniplanar double point are obtained. The complete determination of the cubic surfaces having a degenerate Sylvester pentahedron is due to C. Rodenberg, 'Zur Klassifikation der Flächen dritter Ordnung', *Math. Ann.*, vol. 14 (1879), pp. 46–110.

regard to the lines s_{01}, s_{02}, s_{12}, it follows that either the latter coincide in a single line, or the former consist of χ_0 counted twice and an arbitrary plane through S_{012}.

If χ_0 $(=\chi_1=\chi_2)$, χ_3, *and* χ_4 *are* 3 *distinct planes, the limiting net of* N *does not contain any surface of the pencil determined by* χ_3^3 *and* χ_4^3.

In fact all the surfaces of this pencil have as triple line the line s_{34}, which (§ 89) does not go through S_{012}; and, on the other hand, the only surface of the limiting net having a triple line not containing S_{012} is χ_3^0, which certainly does not belong to the pencil.

It follows that the join of the net and pencil just considered is the linear system Θ; and, since Θ contains the non-singular surface G, the net cannot contain χ_0 as a fixed double component, so that s_{01}, s_{02}, s_{12} must coincide into a single line, nor can s_{34} be incident with this line. The surface G therefore belongs to a pencil determined by a surface of the net, composed of χ_0 and of a quadric cone K touching χ_0 along s_{12} $(=s_{01}=s_{02})$ and by a surface composed of 3 distinct planes through s_{34}; moreover, s_{34} intersects χ_0 in a well-defined point $(S_{034}=S_{134}=S_{234})$, which does not belong to K, and has consequently a well-determined polar plane ρ with respect to K, not going through it. The 4 planes χ_0, χ_3, χ_4, ρ are independent, so that we can assume them to be fundamental planes of the coordinates, $x_0=0$, $x_1=0$, $x_2=0$, $x_3=0$ respectively. With a further convenient choice of the unit point, and on the hypothesis that the surface G is non-cyclic, we obtain for this surface the *canonical equation*

$$2\lambda x_0^3+x_1^3+x_2^3-3x_0(\mu x_0x_1+x_0x_2+x_3^2)=0 \quad (\text{with } \mu\neq 0), \tag{9}$$

and the condition $$\lambda+\mu^{\frac{3}{2}}\pm 1\neq 0 \tag{10}$$

expressing its non-singularity; the surfaces G under consideration depend on 17 parameters.

The Hessian of (9) is given by the equation

$$x_0x_1x_2(-2\lambda x_0+\mu x_1+x_2)+x_0^3(x_1+\mu^2x_2)-x_1x_2x_3^2=0;$$

and a simple calculation shows that, by virtue of (10), it has only 4 singular points

$$B(0,1,-\mu,0),\qquad A_1(0,1,0,0),\qquad A_2(0,0,1,0),\qquad A_3(0,0,0,1),$$

the first of which is double conic, while the 3 others are double biplanar having the tangent cones $x_0x_2=0$, $x_0x_1=0$, $x_1x_2=0$ respectively. The points A_1, A_2, B are evidently on a line $(x_0=x_3=0)$, lying on the Hessian, which has along it the plane $(x_0=0)$ containing its 4 singular points as a fixed tangent plane. These properties of the

Hessian show that the surface (9) is neither generic nor cyclic, and that it also differs from the surfaces considered in § 91 (i); we can say in conclusion that:

A non-singular cubic surface (9) *has no Sylvester representation; when any generic surface tends to it, the* 5 *faces of the Sylvester pentahedron of the former tend to the plane* $x_0 = 0$ *counted* 3 *times and to the planes* $x_1 = 0$, $x_2 = 0$: *these three planes are connected with the singular points of the Hessian surface of* (9) *in the manner explained above.*

94. We shall now prove that, with the assumptions of § 89, and supposing G non-cyclic, *none of the further* a priori *possible coincidences among the* 5 *planes* χ_i *can occur*, namely, that 3 at least of these planes must be distinct. For this purpose we consider in succession the three imaginable cases in which these planes reduce to 2 distinct planes or to a single plane, and show their incompatibility with the non-singularity of the surface G to which § 89 refers.

(i) *Let us suppose* $\chi_0 = \chi_1 = \chi_2$, $\chi_3 = \chi_4$, $\chi_0 \neq \chi_3$. Then, by means of an argument similar to one explained in § 93, we see that the limits of the net determined by π_0^3, π_1^3, π_2^3, and of the pencil determined by π_3^3, π_4^3, have no surface in common; so that G belongs to the join of this limiting net and pencil. But, since all the surfaces of the former have s_{01} as a double line (§ 93), and all the surfaces of the latter have χ_3 as at least a double component (§ 91), G must be singular, which contradicts our hypothesis.

(ii) *Let us now suppose* $\chi_0 \neq \chi_1 = \chi_2 = \chi_3 = \chi_4$. In view of § 89 we have consequently $s_{01} = s_{02} = s_{03} = s_{04}$, all these lines coinciding with the intersection s of χ_0 and χ_1; but

$$s_{12} \neq s_{34}, \qquad s_{13} \neq s_{42}, \qquad s_{14} \neq s_{23}, \tag{11}$$

and none of the points S_{ijl} ($i, j, l = 1, 2, 3, 4$) can belong to s. We consider the ∞^3 linear system Ξ determined by π_1^3, π_2^3, π_3^3, π_4^3, and its limit Ω; since all the polar quadrics of any surface of Ξ have $\pi_1 \pi_2 \pi_3 \pi_4$ as a self-polar tetrahedron, it follows that all the polar quadrics Q of any surface of Ω have the three pairs of lines appearing in (11) as conjugate, and the 4 (not necessarily distinct) points S_{ijl} ($i, j, l = 1, 2, 3, 4$) are conjugate, with respect to them, to all the points of χ_1. Each surface of Ω must therefore go at least twice through each of these 4 points, so that none of the former can coincide with χ_0^3, and Θ is the joining system of Ω and χ_0^3. As this system contains the non-singular surface G, it is impossible that all the surfaces of Ω have a double line, so that the 4 points S_{ijl} must coincide in a single point, S say: this cannot be

a triple base point of Ω, since otherwise all the surfaces of Θ would be cyclic, and all the surfaces of Ω have therefore in S a double uniplanar point, with χ_1 as tangent plane in it.

The three pairs of lines appearing in (11) belong to the pencil of centre S; should two of them be distinct, then all the quadrics Q would go through (at least) one and the same line of this pencil, which would be double for all the surfaces of Ω. Since this is impossible, we must, for instance, have

$$s_{12} = s_{13} = s_{14} = p, \qquad s_{34} = s_{42} = s_{23} = q, \qquad p \neq q.$$

On every generic line of π_1 the surfaces of Ξ intersect sets of three points belonging to the g_3^2 determined by the triple points intersected on it by π_2^3, π_3^3, π_4^3. These points in the limit coincide into one and the same point of p; hence, on account of § 90, p must be a base line of Ω. Since the distinct lines p, q must be conjugate with respect to all the quadrics Q defined above, it follows that all the surfaces of Ω must intersect χ_1 along p counted three times. Introducing in [3] projective homogeneous coordinates (x_0, x_1, x_2, x_3) such that S is the fundamental point $(1, 0, 0, 0)$, χ_1 is the fundamental plane $x_1 = 0$, and p is the fundamental line $x_1 = x_2 = 0$, we see that the equation of every surface of Ω must be of the form

$$x_1^2 x_0 + a x_2^3 + x_1 \sum_{i=1}^{3} \sum_{j=1}^{3} b_{ij} x_i x_j = 0. \tag{12}$$

The Hessian of such a surface (if it is not indeterminate) must reduce to the plane $x_1 = 0$ counted 4 times, since the Hessian of each surface of Ξ is the tetrahedron $\pi_1 \pi_2 \pi_3 \pi_4$; therefore in (12) we must have either $b_{33} = 0$ or $a = 0$, and in both cases we arrive at a contradiction: in the first case, in fact, p would be a double line of the surface (12), while in the second case each surface of Ω would have χ_1 as a component, so that each surface of Θ would have some singular point on s.

(iii) *Finally let us suppose* $\chi_0 = \chi_1 = \chi_2 = \chi_3 = \chi_4$. If i, j, l, m, n are the numbers 0, 1, 2, 3, 4 taken in any arrangement, we have (§ 89) that the point S_{ijl} does not belong to the line s_{mn} and is conjugate to this line with respect to every polar quadric of G; moreover, s_{ij} and s_{mn} are distinct and, since the polar quadrics cannot have a base-point (as G is non-singular), the latter must intersect on χ_0 the whole net of conics having the (proper) triangle s_{12}, s_{34}, $S_{125} S_{345}$ as self-polar. It follows that each pair such as S_{ijl}, s_{mn} consists of a vertex of this triangle and its opposite side; hence each of the 10 lines s_{mn} coincides with one of the 3 sides of the triangle, so that there is certainly at least one of the

latter which coincides with at least 4 of the former. These lines, taken 2 by 2, must have an index in common; they have therefore to be the 4 lines s_{mn} for which one of the indices assumes one and the same value, for instance the value 0. Each of the other 6 lines s_{mn} must coincide with one of the remaining sides of the triangle considered above, in such a way that (11) hold. With our assumptions, and by a proper choice of the names of the 5 planes π_i, we have consequently left the one possibility expressed by the relations

$$s_{01} = s_{02} = s_{03} = s_{04}, \qquad s_{12} = s_{13} = s_{14}, \qquad s_{34} = s_{42} = s_{23},$$

from the last of which it follows that

$$S_{012} = S_{013} = S_{014}.$$

Upon the line r_{01}, the surfaces of Σ intersect sets of the g_3^2 determined by the triple points R_{012}, R_{013}, R_{014}, so that, by virtue of § 90, all the surfaces of Θ go through the point S_{012} $(= S_{013} = S_{014})$. Moreover, since the Hessian of F intersects the planes π_0 and π_1 along the quadrilaterals $r_{01}r_{02}r_{03}r_{04}$ and $r_{10}r_{12}r_{13}r_{14}$ (§ 84), which have 2 distinct limits on $\chi_0 = \chi_1$ when $F \to G$, we deduce that the Hessian of G must contain the plane χ_0 as a component.

Let us introduce in [3] homogeneous projective coordinates (x_0, x_1, x_2, x_3), assuming χ_0 as fundamental plane $x_0 = 0$ and, upon it, the triangle considered above as fundamental triangle. Such a triangle having to be self-polar with respect to all the polar quadrics of G, the equation of this surface must be of the form

$$\sum_{i=1}^{3} a_i x_i^3 + x_0 \sum_{i=0}^{3} \sum_{j=0}^{3} b_{ij} x_i x_j = 0;$$

and one at least of the coefficients a_i, for instance a_3, has to be zero, since G goes through the fundamental point S_{012}. If we express the fact that the Hessian surface of G contains the plane χ_0 as part, we see that either at least another of the coefficients a_i is zero, for instance $a_2 = 0$, or $b_{13} = b_{23} = b_{33} = 0$; but both conclusions contradict our assumption of the non-singularity of G, as the last equation then represents a surface having some singular point on the line $x_0 = x_1 = 0$.

The results of the discussion of this paragraph and of §§ 91–3 can be summarized by saying that:

The only non-singular cubic surfaces for which a Sylvester representation does not exist are those representable by the equations (6) *and* (9) *of* §§ 91 *and* 93, *which constitute* 2 *distinct systems of dimension* 18 *and* 17 *respectively. The cyclic surfaces are the only non-singular cubic surfaces for which the Hessian is reducible.*

95. The last part of the final theorem of § 94 suggests the question of determining *all the cubic surfaces having a reducible Hessian surface.* If we exclude the *cones*, for which the Hessian is indeterminate, and the non-singular *cyclic equianharmonic* or *non-equianharmonic surfaces*, for which (§ 88) the respective Hessians consist of 4 independent planes and of a plane and a cubic cone without multiple generators, we can limit ourselves to considering a cubic surface F with (at least) one double point P, and to determining in which cases its Hessian H is reducible. If we assume P as the fundamental point $(1, 0, 0, 0)$ of the homogeneous projective coordinates $(x_0,\ x_1,\ x_2,\ x_3)$, then F is represented by an equation of the type

$$x_0 f(x_1, x_2, x_3)+\phi(x_1, x_2, x_3) = 0, \tag{13}$$

where f, ϕ are homogeneous polynomials in x_1, x_2, x_3, of degrees 2, 3 respectively, the first and second partial derivates of which we denote by f_i, ϕ_i and by f_{ij}, ϕ_{ij}; and we can distinguish three cases, according as F has at P a conic, or a biplanar, or a uniplanar double point.

In the first case we have

$$c = \begin{vmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{vmatrix} \neq 0, \tag{14}$$

and the equation of H is

$$\begin{vmatrix} 0 & f_1 & f_2 & f_3 \\ f_1 & x_0 f_{11}+\phi_{11} & x_0 f_{12}+\phi_{12} & x_0 f_{13}+\phi_{13} \\ f_2 & x_0 f_{21}+\phi_{21} & x_0 f_{22}+\phi_{22} & x_0 f_{23}+\phi_{23} \\ f_3 & x_0 f_{31}+\phi_{31} & x_0 f_{32}+\phi_{32} & x_0 f_{33}+\phi_{33} \end{vmatrix} = 0.$$

By applying the Euler theorem this equation can be reduced to the form

$$-2cx_0^2 f(x_1, x_2, x_3)-2x_0\Big[\sum_{i=1}^{3}\sum_{j=1}^{3} c_{ij}\phi_{ij} f(x_1, x_2, x_3)-3c\phi(x_1, x_2, x_3)\Big]+$$
$$+\begin{vmatrix} 0 & f_1 & f_2 & f_3 \\ f_1 & \phi_{11} & \phi_{12} & \phi_{13} \\ f_2 & \phi_{21} & \phi_{22} & \phi_{23} \\ f_3 & \phi_{31} & \phi_{32} & \phi_{33} \end{vmatrix} = 0, \tag{15}$$

where c_{ij} indicates the algebraic complement of f_{ij} in the determinant c; and (15) shows that H has at P a conic double point, with the same tangent cone $f(x_1, x_2, x_3) = 0$ as F. It follows that, if H is reducible, it must contain as a component either this quadric cone, or a plane not going through P.

The first possibility, namely, the fact that $f(x_1, x_2, x_3)$ is a factor of the left-hand side of (15), implies that $\phi(x_1, x_2, x_3)$ must be divisible by $f(x_1, x_2, x_3)$; then F breaks up into the quadric cone $f(x_1, x_2, x_3) = 0$ and a plane not containing P: if we therefore assume this plane as fundamental plane $x_0 = 0$, we obtain in (13) $\phi(x_1, x_2, x_3) \equiv 0$, so that (15) reduces to $x_0^2 f(x_1, x_2, x_3) = 0$.

If the second, but not the first, of the two above considered possibilities arises, we can assume the plane component of the Hessian to be $x_0 = 0$, and then the relation

$$\begin{vmatrix} 0 & f_1 & f_2 & f_3 \\ f_1 & \phi_{11} & \phi_{12} & \phi_{13} \\ f_2 & \phi_{21} & \phi_{22} & \phi_{23} \\ f_3 & \phi_{31} & \phi_{32} & \phi_{33} \end{vmatrix} = 0, \tag{16}$$

but not $\phi(x_1, x_2, x_3) = 0$, must hold *identically* with respect to the variables x_1, x_2, x_3. We shall show that *this condition is equivalent to supposing $\phi(x_1, x_2, x_3)$ to be the cube of a linear form.* We remark for this purpose that, if we interpret x_1, x_2, x_3 as projective homogeneous coordinates on a plane, and assume $f(x_1, x_2, x_3) = 0$, $\phi(x_1, x_2, x_3) = 0$ to be the equations of any two plane algebraic curves f, ϕ, of arbitrary orders m, n, then the equation (16) represents in general an algebraic curve of order $2m+2n-6$, covariant of f, ϕ, and locus of the points (x_1, x_2, x_3) for which the polar line with respect to f and the polar conic with respect to ϕ touch each other: we have to prove that, if f is an irreducible conic and ϕ is a cubic, this locus is indeterminate only if ϕ reduces to a line counted three times.†

We begin by establishing that, if (16) holds identically, the point of intersection of the polar lines of every point $M(x_1, x_2, x_3)$ with respect to f and ϕ always belongs to the Hessian of ϕ. If M is generic, these polar lines are first of all certainly distinct, since the distinct curves f, ϕ are respectively the loci of the points M which belong to the former or to the latter of them; we can take M as the fundamental point $(1, 0, 0)$, so that its polar lines are

$$f_1 = 0, \qquad \phi_{11} = 0. \tag{17}$$

† This property—which is similar to a well-known result concerning the case $m = n = 2$—cannot be extended to curves of every order, since the left-hand side of (16) vanishes identically if f, ϕ are, for instance, the non-singular conic and quartic

$$f(x_1, x_2, x_3) \equiv x_1^2+x_2^2+x_3^2 = 0 \quad \text{and} \quad \phi(x_1, x_2, x_3) \equiv a_1 x_1^4+a_2 x_2^4+a_3 x_3^4 = 0,$$

with $$a_2 a_3+a_3 a_1+a_1 a_2 = 0.$$

On the other hand, the identity (16) gives

$$\phi_{11}\begin{vmatrix} 0 & f_2 & f_3 \\ f_2 & \phi_{22} & \phi_{23} \\ f_3 & \phi_{32} & \phi_{33} \end{vmatrix} \equiv -(f_2\phi_{13}-f_3\phi_{12})^2 \pmod{f_1}; \tag{18}$$

it follows that, in the point common to the lines (17), we have

$$f_1 : f_2 : f_3 = \phi_{11} : \phi_{12} : \phi_{13}, \tag{19}$$

and that the line $f_1 = 0$ has on it two intersections with the curve represented by equating to zero the right-hand side, and therefore also the left-hand side, of (18): hence in such a point the determinant appearing in (18) must vanish, and the same is true for the Hessian of $\phi(x_1, x_2, x_3)$, owing to (17), (19), and to the fact that f_1, f_2, f_3 cannot be zero at the same point, by virtue of (14).

If ϕ is not a line counted three times, a variable point of its plane has a variable polar line with respect to it; so that each point of the plane is common to the polar lines of (at least) one point with respect to f and ϕ, as there are ∞^1 first polars of f going through it. From the above established property it follows that the Hessian of $\phi(x_1, x_2, x_3)$ vanishes identically; hence, by virtue of a well-known theorem,† we can make the function $\phi(x_1, x_2, x_3)$ independent of x_3 by means of a convenient change of coordinates. Then (16) shows that the Hessian of this function with respect to x_1, x_2 also vanishes identically, as f_3 cannot vanish identically owing to (14); this proves that $\phi(x_1, x_2, x_3)$ must really be the cube of a linear form: and we have essentially only two cases, according as f is or is not tangent to the line represented by equating to zero this linear form. We can say in conclusion that

The only cubic surfaces possessing (at least) one conic double point, and having a reducible Hessian surface, are the three surfaces represented by the equations

$$x_0(x_1 x_2+x_3^2) = 0, \qquad x_0 x_1 x_2+x_0 x_3^2+x_3^3 = 0, \qquad x_0 x_1 x_2+x_0 x_3^2+x_1^3 = 0, \tag{20}$$

whose Hessians are, respectively,

$$x_0^2(x_1 x_2+x_3^2) = 0, \qquad x_0(x_0 x_1 x_2+x_0 x_3^2+3x_1 x_2 x_3) = 0,$$
$$x_0(x_0 x_1 x_2+x_0 x_3^2-3x_1^3) = 0.‡$$

† Cf. P. Gordan—M. Noether, 'Ueber die algebraischen Formen, deren Hesse'sche Determinante identisch verschwindet', *Math. Ann.*, vol. 10 (1876), pp. 547–68.

‡ The first surface (20) consists of an irreducible cone and a plane not containing its vertex; its Hessian breaks up into the same cone and plane, the latter being counted twice. The second surface (20) is irreducible, and projectively characterized by the

Let us return to the surface F represented by (13); we now suppose that it has a biplanar double point at P, so that we can assume

$$f(x_1, x_2, x_3) \equiv x_1 x_2,$$

and the Hessian H is consequently represented by

$$2x_0 x_1 x_2 \phi_{33} - (x_1^2 \phi_{11} - 2x_1 x_2 \phi_{12} + x_2^2 \phi_{22})\phi_{33} + (x_1 \phi_{13} - x_2 \phi_{23})^2 = 0. \quad (21)$$

If ϕ_{33} is identically zero, F *has a double line* $x_1 = x_2 = 0$, i.e. it is ruled, and H reduces to the *two planes* (*counted doubly*) joining that line to the parabolic lines of F, that is, to the two generators along each of which F has a fixed tangent plane; these generators coincide if $\phi_{113}\phi_{223} = 0$, in which case F is a *Chasles-Cayley ruled cubic surface*† and H consists of *a single plane counted four times*. It is easy to prove that, by means of a convenient change of coordinates, the equation of F can be reduced to the form

$$x_0 x_1^2 + x_2 x_3^2 = 0 \quad (22)$$

in the first case, and to the form

$$x_0^2 x_1 + x_0 x_2 x_3 + x_2^3 = 0 \quad (23)$$

in the second case.

If F is not ruled, (21) shows that H has at P a triplanar triple point, with the tangent planes $x_1 = 0$, $x_2 = 0$, $\phi_{33} = 0$; hence H can only be reducible if *it contains as a component* (*at least*) *one of these three planes*. This, as is not difficult to prove, occurs if, and only if, *either F does not contain the edge of P and intersects each of the two planes touching F at P along three lines of a pencil having that edge as a line of their Hessian set, or one at least of the two planes touching F at P intersects F along a line counted three times* (on which F has therefore another biplanar double point, with the plane just considered as one of the planes touching F at it).

Finally it is immediately seen that H is always reducible if *P is a uniplanar double point* of F, in which case *H breaks up into the tangent plane of F at P counted twice and a quadric cone of vertex P.*

property of having a conic double point, A_0, and two biplanar double points, A_1, A_2, in both of which it is touched by one and the same plane ($x_0 = 0$) not going through the former point; its Hessian breaks up into this plane, and a residual cubic surface of the same type (whose equation reduces to $x_0' x_1 x_2 + x_0' x_3^2 - 3x_3^3 = 0$ by putting $x_0' = x_0 + 3x_3$). The third surface (20) is the limit of the second one, when its two biplanar double points become infinitely near.

† For this nomenclature and the equations (22), (23) given below, cf., for instance, L. Cremona, *Opere matematiche*, vol. ii (Milano, Hoepli, 1915), p. 48 and note (109) at p. 447, or E. Bertini, *Complementi di geometria proiettiva* (Bologna, Zanichelli, 1927), pp. 233, 235.

From our analysis it follows in particular that:

The Hessian of a cubic surface can never consist of two irreducible quadrics.

96. Another interesting question† is that of reducing a given quaternary cubic form to a sum of cubes of linear forms of the minimum possible number, and, in particular, of determining this minimum number for each kind of cubic form. The analogous problem for ternary forms is solved by the following theorem.

The minimum number ν of cubes as the sum of which a given ternary cubic form ϕ can be expressed, can only assume each of the values $\leqslant 5$, according as the plane cubic curve $\mathfrak{C}$ represented by the equation $\phi = 0$ belongs to one or the other of the following 5 categories.

(i) $\nu = 1$, if $\mathfrak{C}$ is a line counted three times.

(ii) $\nu = 2$, if $\mathfrak{C}$ consists of three distinct lines of a pencil. In this pencil the Hessian of three such lines consists of two distinct lines, representing two linear forms such that ϕ can be expressed as the sum of their cubes.

(iii) $\nu = 3$, if $\mathfrak{C}$ either is equianharmonic or consists of a line counted twice and a line counted once. In the first case the Hessian of $\mathfrak{C}$ consists of three independent lines, representing three linear forms such that ϕ can be expressed as the sum of their cubes; in the second case ϕ has ∞^2 decompositions into the sum of three cubes, all obtainable by taking a line l through the triple point of $\mathfrak{C}$, distinct from its double line, and applying (ii) to every cubic without a double line of the pencil determined by $\mathfrak{C}$ and the line l counted three times.

(iv) $\nu = 4$, if $\mathfrak{C}$ is either irreducible non-equianharmonic (and possibly singular), or consists of three independent lines or of an irreducible conic and a non-tangent line. In every case ϕ has ∞^2 decompositions into the sum of four cubes, all obtainable by applying (iii) to the equianharmonic curves of the pencils determined by $\mathfrak{C}$ and a generic line of its plane counted three times.‡ It is immediately verified that

† Suggested by H. F. Baker, op. cit. in § 62, vol. iii, p. 212.

‡ Each of those pencils contains a finite and non-zero number of equianharmonic curves, obtainable by equating to zero the absolute invariant J of the curves of the pencil, which gives an algebraic equation for the parameter λ on which such curves may be made to depend. The only possible exception could arise in the case in which J is independent of λ; which, however, can be excluded on our hypothesis either by referring to the classification [given by O. Chisini, 'Sui fasci di cubiche a modulo costante', *Rendic. del Circolo Mat. di Palermo*, vol. 41 (1916), pp. 59–93] of all the pencils of cubics having $J =$ const., or by noticing that each of our pencils has some curve with one, two, or three ordinary double points (that is, for which $J = 1$), without having $J = 1$ identically, since the pencil itself has no double base-point.

each plane quadrilateral whose lines are defined by equating to zero 4 cubes having ϕ as their sum, is inscribed in the Hessian cubic of $\mathfrak{C}$.†

(v) $\nu = 5$, if $\mathfrak{C}$ consists of an irreducible conic and one of its tangent lines. In this case, which is the only one left, ν cannot evidently be less than or equal to 3 and not even equal to 4, since the Hessian of $\mathfrak{C}$—which reduces to the line component of $\mathfrak{C}$ counted three times—does not possess any proper inscribed quadrilateral. The form ϕ can be decomposed in ∞^5 different manners into the sum of 5 cubes, obtainable by considering the ∞^2 pencils determined by $\mathfrak{C}$ and an arbitrary line of its plane counted three times, and by applying (iv) to their generic curve.

97. We can now prove that:

The minimum number ν of cubes as the sum of which a given quaternary cubic form Φ is expressible, can assume each of the values $\leqslant 7$ and only these values, the value $\nu = 7$ implying the vanishing of the discriminant of Φ.

The cubic surfaces F, represented by equating to zero a form Φ having $\nu = 1, 2, 3, 4, 5$, are the projections of the plane cubic curves $\mathfrak{C}$ considered in § 96, in (i), (ii), (iii), (iv), (v) respectively, which includes all the cones; and we have only to add, for $\nu = 4$, the equianharmonic cubic surfaces and, for $\nu = 5$, the cubic surfaces having a well-defined Sylvester pentahedron and the cyclic (possibly singular) cubic surfaces: the corresponding reduction of Φ to a sum of 4 or 5 cubes is connected with the geometric properties explained in §§ 84–8 in the case of the non-singular surfaces.‡

By virtue of the final theorem of § 94, the only non-singular cubic surfaces having $\nu \geqslant 6$ are those representable by the equations (6) and (9) of §§ 91 and 93; and we shall now prove that both of them have $\nu = 6$. This result is obvious for the first of such equations, since—on account of § 96, (iii)—the last term of its left-hand side can be expressed as the sum of three cubes in ∞^2 different ways, one of which is, for instance, given by the identity

$$x_0^2(\lambda_0 x_0+3\lambda_1 x_1+3\lambda_2 x_2+3\lambda_3 x_3) = [(\tfrac{1}{6}\lambda_0+1)x_0+\tfrac{1}{2}\lambda_1 x_1+\tfrac{1}{2}\lambda_2 x_2+\tfrac{1}{2}\lambda_3 x_3]^3+$$
$$+[(\tfrac{1}{6}\lambda_0-1)x_0+\tfrac{1}{2}\lambda_1 x_1+\tfrac{1}{2}\lambda_2 x_2+\tfrac{1}{2}\lambda_3 x_3]^3-$$
$$-2[\tfrac{1}{6}\lambda_0 x_0+\tfrac{1}{2}\lambda_1 x_1+\tfrac{1}{2}\lambda_2 x_2+\tfrac{1}{2}\lambda_3 x_3]^3.$$

† The quadrilaterals can also be introduced in connexion with the envelopes of class 2 apolar to $\mathfrak{C}$; cf., for instance, F. Enriques—O. Chisini, *Lezioni*, cit. in § 62, vol. ii, p. 70.

‡ The covariant of the 5th degree of Φ representing the Sylvester pentahedron of F has been determined by P. Gordan, 'Ueber das Pentaeder der Flächen dritter Ordnung', *Math. Ann.*, vol. 5 (1872), pp. 341–77.

As for (9), we notice that this equation can be written in the form

$$\{x_1^3+x_2^3+t(\mu x_1+x_2)^3\}+\{2\lambda x_0^3-3x_0(\mu x_0 x_1+x_0 x_2+x_3^2)-t(\mu x_1+x_2)^3\} = 0;$$

and that, choosing t generically ($\neq 0$), we can express the quantic in the first bracket as a sum of 2 cubes (§ 96, ii) and the quantic in the second bracket as a sum of 4 cubes (§ 96, iv), since, by equating to zero these expressions, we obtain the equations of 3 distinct planes of a pencil and, respectively, of a cubic cone [of vertex $(0, 1, -\mu, 0)$] without multiple generators.

Let us now suppose F to be singular, and possessing a finite number of double points: then we must certainly have $\nu \leqslant 7$; in fact $\Phi-\alpha^3$ has a non-zero discriminant, if α is a generic linear form, and can therefore be expressed as the sum of at most 6 cubes, by virtue of what we have seen above.

We shall now show that the same limitation $\nu \leqslant 7$ holds also if F has infinitely many singular points; leaving out the cones, which we have already considered, F can only be either an irreducible surface with a double line, and therefore (§ 95) representable by one of the equations (22), (23), or a reducible surface consisting of an irreducible (possibly singular) quadric and a non-tangent plane, or of a non-singular quadric and a tangent plane. The left-hand sides of (22), (23) can be expressed as sums of 6 or, respectively, 7 cubes, since, taking into account § 96, (iii), (iv), each of the terms $x_0 x_1^2$, $x_2 x_3^2$, $x_0^2 x_1$ can be written as the sum of 3 cubes, and $x_2(x_0 x_3+x_2^2)$ as the sum of 4 cubes. If F breaks up into an irreducible quadric and a non-tangent plane, by choosing convenient coordinates we can take $\Phi \equiv x_0(ax_0^2+x_1 x_2+x_3^2)$ (where $a = 0$ if and only if the quadric is singular); this expression can be written as the sum of the forms $x_0(px_0^2+x_1 x_2)$ and $x_0(qx_0^2+x_3^2)$ with $p+q = a$, $pq \neq 0$, expressible as sums of 4 and 2 cubes respectively on account of § 96, (iv), (ii). Finally, if F breaks up into a non-singular quadric and a tangent plane, we can assume

$$\Phi \equiv 6x_1(x_0 x_1+4x_2 x_3) \equiv (x_0+x_1)^3+(x_0-x_1)^3-2x_0^3+(x_1+x_2+x_3)^3+$$
$$+(x_1-x_2-x_3)^3+(-x_1+x_2-x_3)^3+(-x_1-x_2+x_3)^3.$$

Leaving it to the reader to specify exactly the cases in which $\nu = 6, 7$, we show that *the last surface actually has* $\nu = 7$, by proving that Φ cannot be expressed as a sum of $\nu \leqslant 6$ cubes. Let us, in fact, suppose identically

$$3x_0 x_1^2+12x_1 x_2 x_3 \equiv \alpha_1^3+\alpha_2^3+\alpha_3^3+\alpha_4^3+\alpha_5^3+\alpha_6^3, \tag{24}$$

where the α_i's are linear forms in x_0, x_1, x_2, x_3, in some of which, for

instance in $\alpha_6 \equiv a_0 x_0 + a_1 x_1 + a_2 x_2 + a_3 x_3$, x_0 must appear with a coefficient $a_0 \neq 0$. The Hessian H of the cubic surface G represented by the equation

$$3x_0 x_1^2 + 12x_1 x_2 x_3 - \alpha_6^3 = 0$$

breaks up into the plane $x_1 = 0$ and the surface H' of equation

$$x_1^3 - \alpha_6[a_0^2(4x_2 x_3 - x_0 x_1) + a_2 a_3 x_1^2 + 2a_0 x_1(a_1 x_1 - a_2 x_2 - a_3 x_3)] = 0,$$

which intersect along three non-concurrent lines

$$x_1 = x_2 = 0, \qquad x_1 = x_3 = 0, \qquad x_1 = \alpha_6 = 0;$$

since the first two of these lines have in common a simple point of H', it follows that this cubic surface can have no triple point. Whence the impossibility of the identity (24), that is, of G being representable by an equation of the type

$$\alpha_1^3 + \alpha_2^3 + \alpha_3^3 + \alpha_4^3 + \alpha_5^3 = 0.$$

For, were the representation possible, $\alpha_i = 0$ $(i = 1,..., 5)$ could not be the equations of 5 planes 4 by 4 linearly independent, since otherwise the line $x_1 = \alpha_6 = 0$, which is simple for G, could not be double for H (§ 84); and, on the other hand, 4 of these 5 planes cannot be linearly dependent, since otherwise G would be either a cyclic (possibly singular) cubic surface or a cone, and H would consequently break up into a plane and a cubic cone or be indeterminate.†

XIV. Autoprojective non-singular cubic surfaces

98. The polar quadric Q of any simple point P of a cubic surface F with respect to F can, as is well known, be defined as the locus of the harmonic conjugate of P with respect to the pairs of points which are further intersections of F with the lines through P; the quadric Q goes simply through P, having at this point the same tangent plane π as F. It follows that, if Q is reducible, it can only consist of π and a plane χ not containing P; this, on the other hand, can only happen if every line touching F at P is a principal tangent of F, which is equivalent to saying that π intersects F along 3 lines through P, that is, P is an Eckardt point of F. On this hypothesis, χ is called the *harmonic plane* of P as, evidently, the harmonic homology of centre P and fundamental plane χ transforms F into itself. Conversely, if a non-singular cubic surface F has a harmonic homology into itself, the centre P of the homology must belong to F and the polar quadric of P with respect

† Th. Reye ['Geometrischer Beweis des Sylvester'schen Satzes . . .', *Journ. f. die reine u. ang. Math.*, vol. 78 (1874), pp. 114–22] has shown, by means of mechanical considerations, how the general quaternary cubic form can be reduced to a sum of 6 cubes.

to F is reducible (since it contains as a component the fundamental plane of the homology), so that P is an Eckardt point of F. Hence:

A non-singular cubic surface F has as many harmonic homologies into itself as it has Eckardt points, each of the former having as centre one of the latter, and conversely. An Eckardt point P of F (which, in conformity with § 71, *can also be defined as a point of F which is double for its Hessian) is one of the parabolic points of each of the* 3 *lines of F through it: the other parabolic points are* 3 *points of a line* (§ 6) *lying in the harmonic plane of P; this plane intersects F along a cubic curve which, together with the* 3 *lines, constitutes the apparent contour of F from P.*

99. The locus $\mathfrak{L}$ of the poles of a line r of F with respect to the conics which are further intersections of F with the planes through r, meets in a single variable point each of these planes, and has evidently in common with r only the two parabolic points R_1, R_2 of this line, in each of which $\mathfrak{L}$ is simply touched by its parabolic plane. $\mathfrak{L}$ is therefore a skew cubic curve touching F at R_1, R_2 and intersecting the surface further in the double points of the 5 conics among those considered above which are singular, namely, in the 5 k-points associated with r (§ 9), which we may call the points of contact of the 5 tritangent planes of F through r. It is immediately seen that $\mathfrak{L}$ contains as a component a line through R_1 (or R_2) if, and only if, this point is an Eckardt point of F, whence, taking also into account § 9, we deduce that:

The necessary and sufficient condition that a line r of F contains just one Eckardt point P of the surface is that 4 *of the tritangent planes through r touch F at* 4 *coplanar points not lying on r;* in such a case, the plane of these 4 points is the harmonic plane of P, and there is a conic through these 4 points which touches F at the parabolic point of r distinct from P. On the plane cubic curve which is the apparent contour of F from any Eckardt point P there are 3 sets of 4 points, in each of which the tangent plane contains one of the 3 lines of F through P; and *every line joining a point of one set with a point of another set goes through a point of the remaining set* (§ 9).

The necessary and sufficient condition that a line r of F contains 2 *Eckardt points of the surface is that* 3 *of the tritangent planes through r touch F at* 3 *collinear points not lying on r;* in such a case the line containing these 3 points, which we call the *harmonic line* of r, and the line r itself are the axes of a *harmonic biaxial homography transforming F into itself.* Conversely, if F has a harmonic biaxial homography into itself, one of the two axes lies on F and contains two Eckardt points

of the surface, and the other axis is the harmonic line of the former one.

The line joining two Eckardt points of a cubic surface F either lies on F or intersects the surface further in another Eckardt point, according as the harmonic homologies transforming F into itself and having the former two points as centres are, respectively, permutable or non-permutable. In the first case, the product of these two homologies is the harmonic biaxial homography having as axes the line of F which joins the two Eckardt points and its harmonic line; in the second case, the new Eckardt point is the centre of a harmonic homology inherent to F, which transforms each of the two homologies considered above into the other.

100. We propose now to determine *all the non-singular cubic surfaces having some non-identical homographic transformation into themselves,* and also to study for each type of surface the *group* $\mathfrak{K}$ formed by the totality of such transformations.† By virtue of §§ 98, 99 the question includes the study of the non-singular cubic surfaces possessing some *Eckardt point* (which has already been partially done by Eckardt himself ‡), and it can obviously be considered both in the complex and in the real field.

We begin by considering the matter *in the complex field,* and by referring ourselves to a *generic* cubic surface F (§ 89): this has then a well-defined Sylvester pentahedron, given by 5 planes π_i ($i = 0, 1, 2, 3, 4$) 4 by 4 linearly independent; we can—as in § 84—represent these planes and F respectively by the equations $x_i = 0$ connected by (2), and by the equation (1), and, moreover, we denote by A_{ij} the vertex of the pentahedron which is the intersection of the three planes π of indices different from i, j, by α_{ij} the diagonal plane $x_i+x_j = 0$ through the vertex A_{ij} and the opposite side, and by β_{ij} the plane $x_i = x_j$ harmonic conjugate of α_{ij} with respect to the two faces π_i, π_j of the pentahedron intersecting on it.

A homography of F into itself must also transform into itself the Sylvester pentahedron of F, and is therefore represented by a substitution among the x_i's transforming (1) into itself; we shall show further that *the left-hand side of this equation must be transformed into itself by such a substitution,* that is, the constant ω by which it is multiplied when we perform the substitution must be $= 1$. If, in fact, we consider

† $\mathfrak{K}$ is evidently, in every case, a subgroup of the group $\mathfrak{G}$ defined in § 12.
‡ Loc. cit. in § 91.

any set of $n \leqslant 5$ among the x_i's upon which the substitution operates cyclically, we see that $\omega^n = 1$; hence we can have $\omega \neq 1$ only if $n = 5$, i.e. if such a substitution reduces to a single cycle, $(x_4 x_3 x_2 x_1 x_0)$ say, and ω is a primitive root of unity of index 5: but, in this case, the equation (1) necessarily reduces to

$$x_0^3+\omega x_1^3+\omega^2 x_2^3+\omega^3 x_3^3+\omega^4 x_4^3 = 0,$$

and represents a singular cubic surface (§ 109).

It follows that, in order to have some non-identical substitution among the x_i's transforming the non-singular surface (1) into itself, the 5 λ_i's must be not all distinct; and, conversely, if there are some coincidences among the λ_i's, some non-identical substitution of such a type exists and the group $\mathfrak{K}$ can be determined at once. Hence, by a suitable choice of notation for the 5 planes π_i, we obtain only the following 6 possibilities, in which we suppose that the equalities among the λ_i's explicitly stated are the only ones existing.

(i) $\lambda_3 = \lambda_4$. $\mathfrak{K}$ has a single non-identical projective transformation, which is the *harmonic homology* represented by interchanging x_3 and x_4. F has consequently only one Eckardt point, A_{34}, with the tangent plane α_{34} and the harmonic plane β_{34}.

(ii) $\lambda_1 = \lambda_2$, $\lambda_3 = \lambda_4$. $\mathfrak{K}$ is *trirectangular*, and contains the permutable harmonic homologies of centres A_{12}, A_{34} and fundamental planes β_{12}, β_{34}, and their product, the harmonic biaxial homography of axes $A_{12}A_{34}$, $\beta_{12}\beta_{34}$. F has consequently 2 and only 2 Eckardt points, A_{12}, A_{34}, belonging to one of its lines.

(iii) $\lambda_0 = \lambda_1 = \lambda_2$. $\mathfrak{K}$ is simply isomorphic with a *symmetric group of degree* 3, and contains 3 harmonic homologies (2 by 2 non-permutable) and 2 cyclic homographies (not homologies) of period 3. The Eckardt points of F are the collinear points A_{01}, A_{02}, A_{12}.

(iv) $\lambda_0 = \lambda_1 = \lambda_2$, $\lambda_3 = \lambda_4$. $\mathfrak{K}$ is simply isomorphic with the *direct product of two symmetric groups of degrees* 3 *and* 2, and contains 4 harmonic homologies, 3 of which are non-permutable with each other but permutable with the 4th. The Eckardt points of F are the collinear points A_{01}, A_{02}, A_{12} and the point A_{34}, which is joined to the former by 3 lines lying on F.

(v) $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4$. $\mathfrak{K}$ is simply isomorphic with a *symmetric group of degree* 4, and contains 3 pairs of harmonic homologies, any 2 of which are permutable or non-permutable, according as they belong to the same pair or to 2 different pairs. The Eckardt points of F are the 6 vertices of the quadrilateral intersected on π_0 by the other 4 faces

of the pentahedron; and the 3 diagonal lines of such a quadrilateral belong to F.†

(vi) $\lambda_0 = \lambda_1 = \lambda_2 = \lambda_3 = \lambda_4$. $\mathfrak{K}$ is simply isomorphic with a *symmetric group of degree* 5. F is the so-called *diagonal surface*, and has 10 Eckardt points at the 10 vertices of the pentahedron, of which (in accordance with the final result of § 99) it contains the 15 diagonals.‡

(vii) Let us now consider the case in which F is *equianharmonic*, and therefore (§ 88) representable by the equation (4). Every homography of F into itself must also leave unaltered its Hessian, namely, the fundamental tetrahedron; hence $\mathfrak{K}$ is, in this case, *of order* $3^3.4! = 648$, its transformations being represented by substituting for the coordinates x_0, x_1, x_2, x_3 the same quantities taken in any order and multiplied by arbitrary cube roots of unity; $\mathfrak{K}$ contains 18 harmonic homologies, represented by substituting for two of the coordinates, x_i, x_j say, ϵx_j, $\epsilon^2 x_i$ respectively, where ϵ is any cube root of unity: and *these* 18 *harmonic homologies generate by multiplication the whole group* $\mathfrak{K}$.|| F has consequently 18 Eckardt points, which are the intersections of the surface with the edges of its fundamental tetrahedron, its 27 lines being those joining the pairs of such points which belong to opposite sides of this tetrahedron.

If the non-singular cubic surface F is *cyclic*, but non-equianharmonic, every homography of F into itself must leave unaltered both its centre and its fundamental plane, as the former is the only triple point of the Hessian H of F and the latter is the only plane component of H (§ 88). We can represent F by the canonical equation (5) of § 88, and denote by $\mathfrak{C}$ the section of F with its fundamental plane $x_0 = 0$; each homographic transformation of F into itself is then representable by a linear

† These surfaces are studied at length by F. E. Eckardt, loc. cit., §§ 13–22; cf. also § 107 of the present work.

‡ The name of diagonal surface, introduced by A. Clebsch ['Ueber die Anwendung der quadratischen Substitution auf die Gleichungen 5-ten Grades und die geometrische Theorie des ebenen Fünfseits', *Math. Ann.*, vol. 4 (1871), pp. 284–345, § 16], derives from the last property. Later on, in § 102, we give some additional properties of such surfaces.

|| By adopting an obvious symbolism we have in fact

$$\begin{pmatrix} x_i, x_j \\ \epsilon^2 x_j, \epsilon x_i \end{pmatrix}\begin{pmatrix} x_i, x_j \\ \epsilon x_j, \epsilon^2 x_i \end{pmatrix} = \begin{pmatrix} x_i, x_j \\ \epsilon x_i, \epsilon^2 x_j \end{pmatrix};$$

so that, by means of products of these harmonic homologies, we can generate in succession all the homographies representable by multiplying the x_i's with arbitrary roots of unity, all those representable by interchanging two of the x_i's, all those representable by an arbitrary substitution upon the x_i's, and finally all the transformations of $\mathfrak{K}$ itself.

substitution operating separately upon x_0 and x_1, x_2, x_3, which multiplies x_0 by any cubic root of unity and transforms the form

$$x_1^3+x_2^3+x_3^3-3\lambda x_1 x_2 x_3$$

(and therefore also the cubic $\mathfrak{C}$) into itself. We deduce $\mathfrak{K}$ immediately by remembering the projective transformations of a plane cubic into itself,† and we obtain that:

(viii) If $\mathfrak{C}$ *is neither harmonic nor equianharmonic*, then $\mathfrak{K}$ is *of order* 54, all its operations being represented by substituting for x_0, x_1, x_2, x_3 respectively $\epsilon_1 x_0$, x_{i_1}, $\epsilon_2 x_{i_2}$, $\epsilon_2^{-1} x_{i_3}$, where x_{i_1}, x_{i_2}, x_{i_3} is any permutation of x_1, x_2, x_3, and ϵ_1, ϵ_2 are arbitrary cube roots of unity. $\mathfrak{K}$ is therefore the direct product of the cyclic group of order 3 determined by the cyclic homologies of period 3 transforming F into itself (§ 85), and a group of order 18, consisting of the identity, 9 harmonic homologies having the centre in one of the 9 points of inflexion of $\mathfrak{C}$, and 8 cyclic homographies (not homologies) of period 3.

(ix) If $\mathfrak{C}$ *is harmonic*, namely, if λ satisfies the condition

$$\lambda^6-20\lambda^3-8 = 0,‡$$

then $\mathfrak{K}$ is the direct product—*of order* 108—of the cyclic group of order 3, considered above, and a group of order 36, consisting of the 18 above-mentioned operations of the general case and 18 cyclic homographies (not homologies) of period 4.

Hence, in both cases (viii) and (ix), *the Eckardt points F are the* 9 *points of inflexion of the plane cubic* $\mathfrak{C}$; in each of these the tangent plane of F is the join of the tangent line of $\mathfrak{C}$ and of the centre of F, the 27 lines of F being therefore the intersections of F with the 9 planes thus defined.

If, finally, F *is neither generic nor cyclic*, by virtue of § 94 it must be a surface representable by the equation (6) or (9); on account of §§ 91 and 93, a homography of F into itself must in the first case transform into themselves both the plane $x_0 = 0$ and the trihedron $x_1 x_2 x_3 = 0$, and in the second case must transform into itself each of the planes $x_0 = 0$, $x_3 = 0$ and the pair of planes $x_1 x_2 = 0$. A simple consideration shows that the autoprojective cubic surfaces of such types are the following.

(x) If F is represented by the equation (6_1) with $\lambda_1^3 = \lambda_2^3 \neq \lambda_3^3$, we can always reduce to the case $\lambda_1 = \lambda_2 \neq \lambda_3$ by performing the change

† Cf., for instance, F. Enriques—O. Chisini, op. cit. in § 62, vol. ii, p. 231.
‡ Ibid., p. 220.

of coordinates which leaves x_0, x_1, x_3 unaltered and substitutes $\lambda_1/\lambda_2 . x_2$ for x_2. On this hypothesis, F has a single non-identical projective transformation, which is the *harmonic homology* represented by interchanging x_1 and x_2, and a single Eckardt point at $B_3(0, 1, -1, 0)$.

(xi) If F is represented by the equation (6_1) with $\lambda_1^3 = \lambda_2^3 = \lambda_3^3$, we can suppose without restriction $\lambda_1 = \lambda_2 = \lambda_3$. Then $\mathfrak{K}$ is simply isomorphic with a *symmetric group of degree* 3, and contains 3 harmonic homologies 2 by 2 non-permutable and 2 cyclic homographies (not homologies) of period 3. The Eckardt points of F are the 3 collinear points B_1, B_2, B_3.

(xii) If F is represented by the equation (6_2), where λ_1^3, λ_2^3, λ_3^3 have distinct values, then F has a single non-identical projective transformation, which is the *harmonic homology* represented by changing the sign of x_0, and a single Eckardt point, A_0.

(xiii) If F is represented by the equation (6_2) with $\lambda_1^3 = \lambda_2^3 \neq \lambda_3^3$, $\mathfrak{K}$ is *trirectangular* and F has 2 and only 2 Eckardt points, A_0 and B_3, belonging to one of its lines.

(xiv) If F is represented by the equation (6_2) with $\lambda_1^3 = \lambda_2^3 = \lambda_3^3$, $\mathfrak{K}$ is simply isomorphic with the *direct product of two symmetric groups of degrees* 3 *and* 2; the Eckardt points of F are the collinear points B_1, B_2, B_3 and the point A_0, which is joined to the former by 3 lines lying on F.

(xv) If F is represented by the equation (9) with $\lambda(\mu^3-1) \neq 0$, then F has a single non-identical projective transformation, which is the *harmonic homology* represented by changing the sign of x_3, and a single Eckardt point, A_3.

(xvi) If F is represented by the equation (9) with $\lambda = 0$, $\mu^3 \neq 1$, then $\mathfrak{K}$ is a *cyclic group of order* 4, given by the homography

$$x_0' : x_1' : x_2' : x_3' = -x_0 : x_1 : x_2 : ix_3$$

and its powers, and F still has only the one Eckardt point A_3.

(xvii) If F is represented by the equation (9) with $\lambda \neq 0$, $\mu^3 = 1$, then $\mathfrak{K}$ is *trirectangular* and contains 2 permutable harmonic homologies; F has 2 and only 2 Eckardt points, B and A_3, belonging to one of its lines.

We can summarize some of the consequences of our discussion by saying that:

Every autoprojective non-singular cubic surface is always transformed into itself by some harmonic homology, that is, it necessarily contains some Eckardt point. The number of Eckardt points which a non-singular cubic

surface may have is given by the following table, where the corresponding types of our classification and the orders of their groups $\mathfrak{K}$ of homographic transformations are also indicated. In every case the whole group $\mathfrak{K}$ is generated by the product of its harmonic homologies, with the exceptions only of the groups inherent to the surfaces of the types (viii), (ix), *and* (xvi).

Number of Eckardt points	1	2	3	4	6	9	10	18
Type of the cubic surface	(i) (x) (xii) (xv), (xvi)	(ii) (xiii) (xvii)	(iii) (xi)	(iv) (xiv)	(v)	(viii), (ix)	(vi)	(vii)
Order of the group $\mathfrak{K}$	2, 4	4	6	12	24	54, 108	120	648

101. The classification of all the *real non-singular cubic surfaces having some real non-identical homographic transformation into themselves*, and the determination, for each of these surfaces F, of both the *group* $\mathfrak{H}$ of all such transformations and the set of the *real Eckardt points*, is now a very simple matter: all this can, in fact, be deduced without difficulty from the last paragraph. Hence we do not carry out this study completely, but confine ourselves to a few general remarks and the most interesting cases; and we add that, in any case, the *type* of F can be derived from its equation by applying the results of the next section. Such a study is of some interest also in relation to section IX, since:

The homographic transformations of $\mathfrak{H}$ which are positive constitute either the whole group $\mathfrak{H}$ or a self-conjugate subgroup of index 2,† *inducing on the lines of F a group of substitutions which is always a subgroup of the group Γ defined in* § 34, *and which—in special cases—can coincide with* Γ.

We add, moreover, that a real Eckardt point of a real cubic surface F will be called of the 1*st* or 2*nd type*, according as the lines of F

† In the first case no negative homography and, in particular, no (real) harmonic homology can belong to $\mathfrak{H}$, so that F *does not contain any real Eckardt point.* As F is real, the total number of its (complex) Eckardt points must therefore be even. Now, if F has 4 Eckardt points, namely, if it is of the type (iv) or (xiv), 3 of those 4 points lie on a line, which does not contain the 4th Eckardt point, so that at least one of the former and the latter are real. If F has 6 Eckardt points, namely, if it is of the type (v), these 6 points are the vertices of a plane quadrilateral, which is necessarily self-conjugate, so that at least 2 of them are real. And we shall see later on (§§ 102, 103) that F always possesses some real Eckardt point if it is of the type (vi) or (vii). By virtue of the classification of § 100 we have that:

In the real field also, an autoprojective non-singular cubic surface is generally transformed into itself by some real harmonic homology, that is, it contains some real Eckardt point. The only exceptions are the surfaces of the type (ii) *having two (and only two) conjugate complex Eckardt points; each of these has a single real non-identical homographic transformation into itself, which is a harmonic biaxial homography (and is therefore positive).*

concurring in it are all real or one real and 2 conjugate complex (of the 1st kind); the plane touching F at this point is a tritangent plane of the 1st or, respectively, 3rd kind of F. A point of F is an Eckardt point of the 1st or 2nd type if, and only if, it is an *isolated* or, respectively, a *nodal point* of the parabolic curve (§ 71).

A surface containing an Eckardt point of the 2nd type can only be of the types F_3, or F_4, or F_5, since on a surface F_1 or F_2 there are no complex lines of the 1st kind (§ 23). A surface having 2 distinct Eckardt points of the 1st type contains at least 5 distinct real lines, and can therefore only be of the types F_1, or F_2, or F_3 (§ 23). A surface F_3 has only 5 real tritangent planes, which are those through the hyperbolic line of the 2nd kind (§ 31, iii); hence F_3 can only possess one or at the most 2 (real) Eckardt points, necessarily lying on its hyperbolic line of the 2nd kind.

102. A *diagonal surface* F has no projective invariants, since (§ 100, vi) by referring such a surface to its Sylvester pentahedron it can be represented by the equation $x_0^3+x_1^3+x_2^3+x_3^3+x_4^3 = 0$, where the x_i's are connected by the identity (2) of § 84. In the real domain, however, there are 3 cases to distinguish, according as the pentahedron consists (I) of 5 real planes, or (II) of 3 real and 2 conjugate complex planes, or (III) of one real and 2 pairs of conjugate complex planes.

CASE I: $\pi_0, \pi_1, \pi_2, \pi_3, \pi_4$ ARE REAL. The group $\mathfrak{H}$ is represented by the 120 substitutions upon the letters x_0, x_1, x_2, x_3, x_4, the even substitutions giving the positive homographies of $\mathfrak{H}$. The latter constitute, therefore, an icosahedral group; by virtue of section IX *our surface is of the type F_1, and this group induces on the lines of such a surface the whole group Γ_1 inherent to F_1.* Taking into account § 100, we see, moreover, that:

The diagonal surface F_1 has 10 *real Eckardt points* (*all of the* 1*st type*) *at the vertices of its pentahedron; its* 15 *hyperbolic lines coincide with the* 15 *diagonals of the pentahedron, and belong* 3 *by* 3 *to the* 5 *faces and to the* 10 *diagonal planes of the pentahedron: these are, respectively, the* 5 *principal planes* (§ 36) *and the* 10 *non-principal tritangent planes of the* 1*st kind of F_1* (§ 37).

The other 12 (elliptic) lines and 30 tritangent planes (of the 2nd kind) of F_1 can be determined as follows. Each of the latter contains a hyperbolic line, i.e. goes through a diagonal line

$$x_h = x_i+x_j = x_m+x_n = 0$$

(where h, i, j, m, n coincide with 1, 2, 3, 4, 5 apart perhaps from the order); so that its equation can be put in the form

$$\omega x_h+x_i+x_j = 0, \quad \text{equivalent to } \omega' x_h+x_m+x_n = 0 \quad \text{if } \omega+\omega' = 1.$$

A simple calculation shows that such an equation represents a tritangent plane if, and only if,

$$\omega, \omega' = \frac{1\pm\sqrt{5}}{2};$$

and that the 12 elliptic lines of F_1, which (§ 31, i) form a double-six, can be represented (each twice) by the equations

$$x_h+x_i+\omega x_j = 0, \qquad x_h+x_j+\omega' x_m = 0,$$
$$x_h+x_m+\omega x_n = 0, \qquad x_h+x_n+\omega' x_i = 0$$

(only two of which are independent), by permuting arbitrarily the indices i, j, m, n, two lines of the same sextuplet being derived from two permutations of the same class, and conversely.

CASE II: π_0, π_1, π_2 ARE REAL, π_3, π_4 ARE CONJUGATE COMPLEX. The group $\mathfrak{P}$ is represented by the $3!\,2 = 12$ substitutions operating separately upon the letters x_0, x_1, x_2 and x_3, x_4, the even substitutions giving the positive homographies of $\mathfrak{P}$ and forming a group simply isomorphic with a *symmetric group of degree* 3. It is clear that

The diagonal surface F has in the present case 4 real Eckardt points, at the real vertices A_{34} and A_{01}, A_{02}, A_{12} of its pentahedron; the former (which is of the 1st type) is joined to the other 3 (which are collinear and of the 2nd type) by 3 lines lying on F. This surface is of the type F_5, since it contains no other real line and has 13 real tritangent planes, which are π_0, π_1, π_2, the planes touching F at its 4 real Eckardt points, and the 6 planes represented by the equations

$$x_3+x_4+\omega x_i = 0, \qquad x_3+x_4+\omega' x_i = 0 \quad (i = 0, 1, 2).$$

CASE III: π_0 IS REAL, π_1, π_2 AND π_3, π_4 ARE CONJUGATE COMPLEX. The group $\mathfrak{P}$ is represented by the 8 substitutions upon x_1, x_2, x_3, x_4 having x_1, x_2 and x_3, x_4 as imprimitive systems, the even substitutions giving the positive homographies of $\mathfrak{P}$ and constituting a *trirectangular group. This coincides with the group Γ inherent to the surface under consideration, which is of the type F_3* (§ 46); this surface has only *two real Eckardt points*, A_{12} and A_{34}, which both are of the 2nd type and are joined by the hyperbolic line of the 2nd kind of F_3; the only real tritangent planes of F_3 are the 5 tritangent planes through this line, namely, the 2 tritangent planes (of the 1st kind) of equations

$$x_1+x_2+\omega x_0 = 0, \qquad x_1+x_2+\omega' x_0 = 0,$$

the plane π_0 (which is of the 2nd kind), and the two diagonal planes α_{12}, α_{34} touching F_3 at A_{12}, A_{34} (which are of the 3rd kind).

103. An *equianharmonic surface* F has no projective invariants, since (§ 100, vii) it can be represented by the equation (4) of § 88; but in the real domain it presents three cases, which we now consider in succession, according as the 4 faces of its fundamental tetrahedron are all real, or 2 real and 2 conjugate complex, or 2 pairs of conjugate complex planes.

CASE I. Each edge of the fundamental tetrahedron is real, and intersects F in one real and 2 conjugate complex points. From § 100 (vii) it follows that F has 3 real lines, 12 complex lines of the 1st kind, and 12 complex lines of the 2nd kind, so that (§ 23) it is *of the type* F_4. In accordance with § 99, we see, moreover, that:

Our surface F_4 has 6 real Eckardt points, all of the 2nd type, which are the vertices of a plane quadrilateral, and therefore the intersections of the 6 edges of the fundamental tetrahedron with a plane (of equation $x_0+x_1+x_2+x_3 = 0$). The plane and the tetrahedron are invariant under all the transformations of the group $\mathfrak{H}$, which are those represented by the 24 substitutions among the letters x_0, x_1, x_2, x_3.

CASE II. The fundamental tetrahedron has 2 real opposite edges, of which one intersects F in one real and 2 conjugate complex points, and the other intersects F in 3 real points. *The 4 real points just considered are the only real Eckardt points of F; the first one (which is of the 1st type) is joined to the 3 others (which are of the 2nd type) by 3 lines, which are the only real lines of F.* The further complex lines of F are 12 of the 1st kind and 12 of the 2nd kind, so that our surface is still *of the type F_4*; and, taking into account § 100 (vii), we see at once that $\mathfrak{H}$ is the direct product (of order 12) of 2 groups simply isomorphic with 2 symmetric groups of degrees 3 and 2.

CASE III. The group $\mathfrak{H}$ is of order 72, and its positive transformations constitute a self-conjugate subgroup of index 2, direct product of two groups simply isomorphic with two symmetric groups of degree 3. By virtue of section IX *our surface is of the type F_2, and this subgroup induces on the lines of such a surface the whole group Γ_2 inherent to F_2.* Taking into account § 100 (vii), we see, moreover, that:

The equianharmonic surface F_2 has 6 real Eckardt points (all of the 1st type), given by the 2 triplets of points intersected on it by the 2 real edges of its fundamental tetrahedron; each line joining a point of a triplet with a point of the other triplet belongs to F_2, and the 9 lines

which thus we obtain make up the Steiner set of the hyperbolic lines of F_2.

The equation (4) of § 88 represents our surface, if we suppose

$$x_0 = \alpha+i\beta, \qquad x_1 = \alpha-i\beta, \qquad x_2 = \gamma+i\delta, \qquad x_3 = \gamma-i\delta$$

(with α, β, γ, δ real).

That equation, namely, $\alpha^3-3\alpha\beta^2+\gamma^3-3\gamma\delta^2 = 0$, reduces to the symmetric form

$$y_1^3+y_2^3+y_3^3+y_4^3+y_5^3+y_6^3 = 0,$$

by introducing the new real coordinates

$$y_1 = \alpha+\sqrt{3}\beta, \qquad y_2 = \alpha-\sqrt{3}\beta, \qquad y_3 = -2\alpha,$$
$$y_4 = \gamma+\sqrt{3}\delta, \qquad y_5 = \gamma-\sqrt{3}\delta, \qquad y_6 = -2\gamma,$$

connected by the identities

$$y_1+y_2+y_3 = 0, \qquad y_4+y_5+y_6 = 0;$$

with this notation we have immediately a simple real representation of $\mathfrak{P}$, by means of the $2.3!\,3! = 72$ substitutions on the 6 letters y_1, y_2, y_3, y_4, y_5, y_6 having y_1, y_2, y_3 and y_4, y_5, y_6 as imprimitive systems.†

104. The equation (5) of § 88, when the parameter λ varies in the real field, represents a real cyclic cubic surface varying in a pencil, and having the fixed point $(1,0,0,0)$ as its centre and the fixed plane $x_0 = 0$ as its fundamental plane. In this pencil there are only 2 (real) singular surfaces: one (arising from $\lambda = \infty$) degenerates into the trihedron $x_1x_2x_3 = 0$, each edge of which contains 3 distinct base-points—one real and two conjugate complex—of the pencil itself; the other (coming from $\lambda = 1$) is represented by the equation

$$x_0^3+(x_1+x_2+x_3)(x_1+\epsilon x_2+\epsilon^2x_3)(x_1+\epsilon^2x_2+\epsilon x_3) = 0 \quad (\text{with } \epsilon = e^{2\pi i/3}),$$

† The equianharmonic surface F_2 can therefore be considered as the *section of the (real) C. Segre* V_3^3 *of* [4]

$$y_1^3+y_2^3+y_3^3+y_4^3+y_5^3+y_6^3 = 0, \qquad y_1+y_2+y_3+y_4+y_5+y_6 = 0,$$

with one $y_1+y_2+y_3 = y_4+y_5+y_6 = 0$ *of the diagonal primes of its fundamental hexahedron*; and the section of V_3^3 with each face of such a hexahedron is manifestly a diagonal surface F_1. We notice in this connexion that, as V_3^3 has 15 real planes, every one of its non-singular sections with a real prime

$$t_1y_1+t_2y_2+t_3y_3+t_4y_4+t_5y_5+t_6y_6 = 0$$

has at least 15 real lines, and is therefore either of the type F_1 or of the type F_2; it can be shown precisely that, if we denote by θ_i the sum of the products of the 6 t's taken i at a time, and put

$$\Theta = 5\theta_1^4+16\theta_2^2-24\theta_1^2\theta_2+32\theta_1\theta_3-64\theta_4,$$

then *this section is of the type* F_1, *or is of the type* F_2, *or is singular of the type* Φ_1, *respectively, according as* $\Theta > 0$, *or* $\Theta < 0$, *or* $\Theta = 0$.

and has at $(0,1,1,1)$ a real double point whose tangent cone consists of two conjugate complex planes

$$x_1+\epsilon x_2+\epsilon^2 x_3 = 0 \quad\text{and}\quad x_1+\epsilon^2 x_2+\epsilon x_3 = 0.$$

Each of the remaining surfaces can be continuously deformed, within the pencil, into one or the other of these two singular surfaces, remaining non-singular during the deformation: by virtue of either § 30 or § 25, we see that *all the surfaces of our pencil, save the two singular ones just considered, are of the type F_4.*†

This result could also be easily deduced from § 100 (viii), (ix), which shows that:

Every real cyclic non-equianharmonic cubic surface is of the type F_4 and has 3 real Eckardt points (of the 2nd type), which are collinear and coincide with the 3 real points of inflexion of the cubic curve intersected on it by its fundamental plane; the group $\mathfrak{H}$ inherent to such a surface is of order 6, and consists of the identity, the 3 harmonic homologies defined by these 3 points, and 2 homographies (not homologies) of period 3.

105. *A real surface F of the type* (i) *of* § 100 presents two cases, according as the three planes π_0, π_1, π_2 (as well as the corresponding coefficients λ_0, λ_1, λ_2 of the equation (1) of § 84) are all real or one real and two conjugate complex; in each case $\lambda_3 = \lambda_4$ must be real, but π_3 and π_4 can be either real or conjugate complex, and the *only Eckardt point A_{34} of F is real.* For this Eckardt point, the property of being *of the* 1*st* or *of the* 2*nd type* is connected with the sign of the real expression

$$\Delta = \lambda_0^2\lambda_1^2+\lambda_0^2\lambda_2^2+\lambda_1^2\lambda_2^2-2\lambda_0\lambda_1\lambda_2(\lambda_0+\lambda_1+\lambda_2),$$

whose vanishing is equivalent to the fact that the plane α_{34} (of equation $x_3+x_4 = 0$), which touches F at A_{34}, intersects F along three non-distinct lines, and is therefore to be excluded if F is non-singular. The behaviour of the Eckardt point A_{34} in the different cases is given by the following table:

	The planes π_0, π_1, π_2 are all real	*The planes π_0, π_1, π_2 are one real and two conjugate complex*
$\Delta < 0$	A_{34} is of the 1st type	A_{34} is of the 2nd type
$\Delta > 0$	A_{34} is of the 2nd type	A_{34} is of the 1st type

† Of that type are, in particular, the equianharmonic surfaces of the pencil, which is in accordance with § 103, I and II.

106. A *real surface of the type* (iv) *of* § 100, namely, a surface F represented by an equation of the form

$$x_0^3+x_1^3+x_2^3+\lambda(x_3^3+x_4^3) = 0,$$

where the x_i's are connected by the identity (2) of § 84 and λ is real, presents essentially 4 cases corresponding to the different possibilities about the reality of the planes π_i. In each case, when λ varies between $-\infty$ and $+\infty$ the surface F describes a pencil, whose only singular surfaces are those corresponding to the values 0, 4/9, 4, ∞ of the parameter λ (§ 109); these are, respectively, an equianharmonic cone of vertex A_{34}, a surface F' with a single ordinary double point of coordinates $(2,2,2,-3,-3)$, a surface F'' with three distinct ordinary double points of coordinates $(2,-2,-2,1,1)$, $(-2,2,-2,1,1)$, $(-2,-2,2,1,1)$, and a surface composed of three distinct planes of a pencil. By virtue of § 24, the surfaces F arising from values of λ comprised between a consecutive pair of the values $-\infty$, 0, 4/9, 4, $+\infty$ are all of the same type; and the complete classification of the non-singular surfaces F that we thus obtain in the different cases, as well as the determination of their group $\mathfrak{P}$ (defined in § 101) and of their real Eckardt point, are given by the table opposite.

The above statements concerning the group $\mathfrak{P}$ and the real Eckardt points are immediate consequences of §§ 100, 105, and hardly need explanation. We must, however, justify the distinction between the types of our surfaces F. If $4/9 < \lambda < 4$, such a distinction follows at once from § 102, by noticing that F reduces to a diagonal surface for $\lambda = 1$. If either $0 < \lambda < 4/9$ or $\lambda > 4$, the surface F can be deduced from this diagonal surface by means of a continuous variation within a pencil, such that all the intermediate positions are non-singular with the exception of a single surface, F' or F'', which has, respectively, one or three ordinary double points not belonging to the base curve of the pencil. Since the type of the initial diagonal surface is *odd*, we deduce from § 25 that F must be of an *even* type, i.e. either F_2 or F_4: and the choice between these two types follows at once by remembering the remarks of § 101 concerning the real Eckardt points.

If finally $\lambda < 0$, the surface F can be continuously deformed into an equianharmonic surface, by considering the surface (variable within a pencil) represented by the equation

$$\mu x_0^3+x_1^3+x_2^3+\lambda(x_3^3+x_4^3) = 0,$$

when μ describes the interval $1 \geqslant \mu \geqslant 0$; the only intermediate singular surface comes from $\mu = 1/4$, and has two non-base ordinary double

		$\lambda < 0$			$0 < \lambda < 4/9$			$4/9 < \lambda < 4\ (\lambda \neq 1)$			$4 < \lambda$		
		Real Eckardt points		*Type of the cubic surface*	*Real Eckardt points*		*Type of the cubic surface*	*Real Eckardt points*		*Type of the cubic surface*	*Real Eckardt points*		*Type of the cubic surface*
	Group $\mathfrak{G}$	*of the 1st type*	*of the 2nd type*		*of the 1st type*	*of the 2nd type*		*of the 1st type*	*of the 2nd type*		*of the 1st type*	*of the 2nd type*	
$\pi_0, \pi_1, \pi_2, \pi_3, \pi_4$ *all real*	*Of order 12,* repres. by the subst. upon x_0, x_1, x_2 and upon x_3, x_4	A_{34}	A_{01} A_{02} A_{12}	F_4	A_{01} A_{02} A_{12} A_{34}	. .	F_2	A_{01} A_{02} A_{12} A_{34}	. .	F_1	A_{34}	A_{01} A_{02} A_{12}	F_4
π_0, π_1, π_2 *real,* π_3, π_4 *conjugate complex*	*Of order 12,* repres. by the subst. upon x_0, x_1, x_2 and upon x_3, x_4	A_{01} A_{02} A_{12} A_{34}	. .	F_2	A_{34}	A_{01} A_{02} A_{12}	F_4	A_{34}	A_{01} A_{02} A_{12}	F_5	A_{01} A_{02} A_{12} A_{34}	. .	F_2
π_0, π_3, π_4 *real,* $\pi_1\ \pi_2$ *conjugate complex*	*Of order 4,* repres. by the subst. upon x_1, x_2 and upon x_3, x_4	. .	A_{12} A_{34}	F_4	A_{12}	A_{34}	F_4	A_{12}	A_{34}	F_5	. .	A_{12} A_{34}	F_4
π_0 *real,* π_1, π_2 *and* π_3, π_4 *conjugate complex*	*Of order 4,* repres. by the subst. upon x_1, x_2 and upon x_3, x_4	A_{12}	A_{34}	F_4	. .	A_{12} A_{34}	F_4	. .	A_{12} A_{34}	F_3	A_{12}	A_{34}	F_4

points of coordinates $(2, -1, -1, \pm 1/\sqrt{\lambda}, \mp 1/\sqrt{\lambda})$ (which are real or conjugate complex, according as π_3 and π_4 are conjugate complex or real): as the type of every equianharmonic surface is *even* (§ 103), the same can be said of F by virtue of § 25, and also in the present case we reach the conclusions stated on account of § 101.

107. A *real surface of the type* (v) *of* § 100, namely, a surface F represented by an equation of the form

$$\lambda x_0^3 + x_1^3 + x_2^3 + x_3^3 + x_4^3 = 0,$$

where the x_i's are connected by the identity (2) of § 84 and λ is real, presents essentially 3 cases in correspondence with the different possibilities concerning the reality of the planes π_i. In each case, when λ varies between $-\infty$ and $+\infty$ the surface F describes a pencil, whose only singular surfaces are those corresponding to the values 1/16, 1/4, ∞ of the parameter λ (§ 109); these are, respectively, a surface F' with a single ordinary double point of coordinates $(4, -1, -1, -1, -1)$, a surface F'' with 4 distinct ordinary double points of coordinates

$$(2, 1, -1, -1, -1), \qquad (2, -1, 1, -1, -1),$$
$$(2, -1, -1, 1, -1), \qquad (2, -1, -1, -1, 1),$$

and the plane π_0 counted 3 times. By virtue of § 24, the surfaces F arising from values of λ comprised between a consecutive pair of the values $-\infty$, 1/16, 1/4, $+\infty$ are all of the same type; and the complete classification of the non-singular surfaces F that we thus obtain in the different cases, as well as the determination of their group $\mathfrak{H}$ (defined in § 101) and of their real Eckardt points, are given by the table opposite.

The above statements concerning the group $\mathfrak{H}$ and the real Eckardt points are simple consequences of §§ 100, 105; the type of F for $\lambda < 1/16$ or $\lambda > 1/4$ follows at once from §§ 103, 102, considering that such a surface reduces to an equianharmonic or a diagonal one, for $\lambda = 0$ and $\lambda = 1$ respectively. If $1/16 < \lambda < 1/4$, the surface F can be continuously deformed within a pencil into that equianharmonic surface, and the only intermediate singular surface is F', the singular point of which is not a base-point; by remembering § 25, it follows that, according as 5, 3, or only 1 of the planes π_i are real, the type of our surface F can only be either F_3 or F_5, F_3 or F_5, or F_1 or F_3 respectively. In the first case F cannot be of the type F_3, since it has more than 2 real Eckardt points (§ 101), and it is therefore of the type F_5.

In order to complete the discussion in the other two cases (in both

	Group $\mathfrak{G}$	$0 \neq \lambda < 1/16$: Real Eckardt points of the 1st type	$0 \neq \lambda < 1/16$: Real Eckardt points of the 2nd type	$0 \neq \lambda < 1/16$: Type of the cubic surface	$1/16 < \lambda < 1/4$: Real Eckardt points of the 1st type	$1/16 < \lambda < 1/4$: Real Eckardt points of the 2nd type	$1/16 < \lambda < 1/4$: Type of the cubic surface	$1/4 < \lambda \neq 1$: Real Eckardt points of the 1st type	$1/4 < \lambda \neq 1$: Real Eckardt points of the 2nd type	$1/4 < \lambda \neq 1$: Type of the cubic surface
$\pi_0, \pi_1, \pi_2, \pi_3, \pi_4$ *all real*	*Of order 24,* repres. by the subst. upon x_1, x_2, x_3, x_4	. .	$A_{12}\ A_{34}$ $A_{13}\ A_{42}$ $A_{14}\ A_{23}$	F_4	. .	$A_{12}\ A_{34}$ $A_{13}\ A_{42}$ $A_{14}\ A_{23}$	F_5	$A_{12}\ A_{34}$ $A_{13}\ A_{42}$ $A_{14}\ A_{23}$	. .	F_1
π_0, π_1, π_2 *real,* π_3, π_4 *conjugate complex*	*Of order 4,* repres. by the subst. upon x_1, x_2 and upon x_3, x_4	A_{12}	A_{34}	F_4	A_{12}	A_{34}	F_3	A_{34}	A_{12}	F_5
π_0 *real,* π_1, π_2 *and* π_3, π_4 *conjugate complex*	*Of order 8,* repres. by the subst. upon x_1, x_2, x_3, x_4 having x_1, x_2 and x_3, x_4 as imprim. syst.	A_{12} A_{34}	. .	F_2	A_{12} A_{34}	. .	F_3	. .	A_{12} A_{34}	F_3

of which π_3, π_4 are conjugate complex and $4 < 1/\lambda$), we consider the surface (variable within a pencil) represented by the equation

$$\lambda x_0^3+\mu(x_1^3+x_2^3)+x_3^3+x_4^3 = 0,$$

when μ describes the interval $1 \geqslant \mu \geqslant \lambda$. Such a surface starts from F and arrives at a surface of one of the categories studied in § 106, whose type is F_2 or F_4 according as π_1, π_2 are real or conjugate complex; the only singular intermediate positions come from

$$\mu = 4\lambda \quad \text{and} \quad \mu = 4\lambda/(1+2\surd\lambda)^2,$$

the first of which has two conjugate complex ordinary double points of coordinates $(1/\surd\lambda, -1/2\surd\lambda, -1/2\surd\lambda, \pm 1, \mp 1)$, while the second has a single non-base ordinary double point, of coordinates $(1/\surd\lambda, -1/\surd\mu, -1/\surd\mu, 1, 1)$. By virtue of § 25 it follows that F can only be of one of the types F_1 or F_3 in the first case, and of one of the types F_3 or F_5 in the second case, so that F is, in fact, of the type F_3 in both cases.

108. As an application of the preceding developments, we can easily obtain all the non-singular cubic surfaces in no point of which the curvature is positive. The parabolic curve of such a surface F must obviously be reduced (in the real domain) to a finite number of isolated points: these points coincide with the Eckardt points of the 1st type of F (§ 101); and, on the other hand, their number must be equal to that of the branches of the parabolic curve of the general cubic surface of the same type as F, namely (§§ 73, 79, 81, 83), it must be 10, 6, 4, 4, 4 according as the surface is of the type F_1, F_2, F_3, F_4, F_5 respectively. The last three cases cannot occur owing to § 101, and in the first two cases F must be a real surface of one of the categories (v)–(ix) of § 100; hence, taking into account §§ 102, 103, 104, 107, we see that:

The only non-singular real cubic surfaces without elliptic points are the diagonal surface of the type F_1 and the equianharmonic surface of the type F_2, on which the parabolic curve reduces to 10 or, respectively, 6 isolated points.†

XV. The type of a real non-singular cubic surface deduced from its canonical equation

109. We propose in this section to determine the type of a real non-singular cubic surface, knowing its canonical equation. We begin with

† H. G. Zeuthen, in § 3 of his paper quoted in the Preface, incidentally affirms (without proof) that the diagonal surface F_1 is *the only one* which has the property mentioned above.

the generic surfaces, and we set down beforehand some general results concerning the (real or complex, non-singular or singular) cubic surfaces having a well-defined proper Sylvester pentahedron.

Such a surface F can be represented (§ 84) by an equation of the type

$$\sum_{i=0}^{4} \lambda_i x_i^3 = 0 \quad (\text{with } \lambda_0 \lambda_1 \lambda_2 \lambda_3 \lambda_4 \neq 0), \tag{1}$$

where the x_i's are superabundant homogeneous projective coordinates connected by the identity

$$\sum_{i=0}^{4} x_i = 0. \tag{2}$$

The polar plane and quadric with respect to F of a point P, the coordinates p_i of which must satisfy the relation

$$\sum_{i=0}^{4} p_i = 0, \tag{3}$$

are respectively represented by the equations

$$\sum_{i=0}^{4} \lambda_i p_i^2 x_i = 0, \tag{4}$$

$$\sum_{i=0}^{4} \lambda_i p_i x_i^2 = 0; \tag{5}$$

hence the first of them is indeterminate, that is, P is a singular point of F, if and only if (4) coincides with (2), namely, if it is possible to alter the p_i's by a convenient non-zero factor, in such a way that

$$p_i = \pm \lambda_i^{-\frac{1}{2}} \quad (i = 0, 1, 2, 3, 4). \tag{6}$$

In this case all the p_i's (like the λ_i's) are non-zero, and the polar quadric (5) is an irreducible cone.† Taking into account the fact that the p_i's must be connected by (3), we have in conclusion that:

The cubic surface F represented by (1), (2) *is singular if, and only if, the expression*

$$\sum_{i=0}^{4} \pm \lambda_i^{-\frac{1}{2}} \tag{7}$$

vanishes for (at least) one convenient choice of the signs; its singular points can only be ordinary double points (in number 4 at most), and are, in fact, the points having the coordinates (6), *where the signs must be chosen in such a way that* (7) *vanishes.*

† In fact, in the 4-dimensional space in which the x_i's are independent coordinates, the equation (5) represents a non-singular V_3^2, so that the section of this with the prime (2) is certainly irreducible.

110. Let us now consider a real non-singular cubic surface F, represented by (1), (2); in order to obtain its type, we can substitute for F every other particular surface F' of equation

$$\sum_{i=0}^{4} \lambda_i' x_i^3 = 0,$$

with the single assumption that *a continuous variation of the λ_i's into the λ_i''s exists, such that in no stage of the variation does the expression* (7) *vanish*: in particular, by first applying such a variation, we can suppose without restriction that some given function of the λ_i's—independent of (7)—does not vanish.

We distinguish for our purpose 3 cases, according as all 5, or 3, or only 1 of the faces π_i (of equation $x_i = 0$, $i = 0, 1, 2, 3, 4$) of the Sylvester pentahedron are real.

111. Case I: $\pi_0, \pi_1, \pi_2, \pi_3, \pi_4$ are real. We can suppose $\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4$ real. *If these parameters are not all of the same sign, then the surface F is of the type F_4.* In fact, if, for instance, either $\lambda_0 < 0, \lambda_1 > 0, \lambda_2 > 0, \lambda_3 > 0, \lambda_4 > 0$, or $\lambda_1 \leqslant \lambda_0 < 0, \lambda_2 > 0, \lambda_3 > 0, \lambda_4 > 0$, we keep $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ unaltered and reduce λ_0 to 0 by means of a continuous variation in the interval $(\lambda_0, 0)$; this variation is of the sort required in § 110, and leads to a surface which is equianharmonic with 4 real fundamental planes, and therefore (§ 103) of the type F_4.

Let us now consider the case in which *all the λ_i's have the same sign.* By changing if necessary the sign of the left-hand side of (1), we can suppose

$$0 < \lambda_0 \leqslant \lambda_1 \leqslant \lambda_2 \leqslant \lambda_3 \leqslant \lambda_4. \tag{8}$$

Moreover, by remembering the remark of § 110, we can suppose that, in fact, none of the equality signs hold; and that, if we put

$$a_i = 1/|\sqrt{\lambda_i}| \qquad (i = 0, 1, 2, 3, 4), \tag{9}$$

whence

$$a_0 > a_1 > a_2 > a_3 > a_4 > 0, \tag{10}$$

we have

$$a_3+a_4 \neq a_1, \qquad a_3+a_4 \neq a_2.$$

On account of § 109, the condition for F to be non-singular is equivalent to supposing a_0 distinct from each of the following 6 expressions

$$\begin{aligned} A &= a_1+a_2-a_3-a_4, & B_1 &= -a_1+a_2+a_3+a_4, \\ B_2 &= a_1-a_2+a_3+a_4, & B_3 &= a_1+a_2-a_3+a_4, \\ B_4 &= a_1+a_2+a_3-a_4, & C &= a_1+a_2+a_3+a_4. \end{aligned} \tag{11}$$

We shall treat separately the three possibilities in which a_3+a_4 is $> a_1$, or $< a_1$ but $> a_2$, or $< a_2$.

Let us first of all suppose $a_3+a_4 > a_1$; then we have

$$a_0 > A \quad \text{and} \quad 0 < A < B_1 < B_2 < B_3 < B_4 < C,$$

in consequence of (10), (11), and we shall show that:

(i) if $A < a_0 < B_1$, our cubic surface is of the type F_1;
(ii) ,, $B_1 < a_0 < B_2$, ,, ,, ,, ,, F_2;
(iii) ,, $B_2 < a_0 < B_3$, ,, ,, ,, ,, F_3;
(iv) ,, $B_3 < a_0 < B_4$, ,, ,, ,, ,, F_4;
(v) ,, $B_4 < a_0 < C$, ,, ,, ,, ,, F_5;
(vi) ,, $C < a_0$, ,, ,, ,, ,, F_4.

In order to prove (i), we derive with continuity from F a variable cubic surface $F'(t)$, for which we adopt a similar notation with dashes, putting

$$\lambda_i' = 1/a_i'^2, \qquad a_i' = a_i+t(a_0-a_i) \quad (i = 0, 1, 2, 3, 4), \quad \text{for } 0 \leqslant t \leqslant 1.$$

On our hypotheses, for every t of the interval $(0, 1)$ we have

$$A' < a_0' < B_1' \leqslant B_2' \leqslant B_3' \leqslant B_4' < C',$$

so that our continuous deformation has the property required in § 110; since the surface $F'(1)$ is a diagonal surface of the type F_1 (§ 102, I), it follows that F is of the same type F_1.

In the case considered in (v) we proceed as above by putting

$$a_0' = (1+2t)a_0, \quad a_j' = a_j+t(a_0-a_j) \quad (j = 1, 2, 3, 4), \quad \text{for } 0 \leqslant t \leqslant 1.$$

Now, for every t of the interval $(0, 1)$ we have

$$A' < B_1' \leqslant B_2' \leqslant B_3' \leqslant B_4' < a_0' < C',$$

and for $t = 1$ we obtain the surface

$$\tfrac{1}{9}x_0^3+x_1^3+x_2^3+x_3^3+x_4^3 = 0;$$

since this is of the type F_5 (§ 107), it follows that the initial surface F (which corresponds to $t = 0$) is of the same type F_5.

The other cases, (ii), (iii), (iv), (vi), can be deduced from the two already investigated by observing that, if in the equation (1) the λ_i's are connected to the a_i's by means of (9), where a_1, a_2, a_3, a_4 are fixed positive numbers satisfying the limitations stated above and a_0 is variable between a_1 and $+\infty$, then the corresponding surface F varies within a pencil, and becomes singular for the values B_1, B_2, B_3, B_4, C of a_0. Each of the singular surfaces thus obtained has a single ordinary double point, which is not a base-point of the pencil (§ 109); so that, when F crosses such a singular position, its type can only increase or

decrease by unity (§ 25). Whence the truth of all the statements made above follows at once.

Let us in the second place suppose $a_1 > a_3+a_4 > a_2$; then we have

$$a_0 > A \quad \text{and} \quad B_1 < A < B_2 < B_3 < B_4 < C,$$

in consequence of (10), (11), and we shall prove that:

(vii) if $A < a_0 < B_2$, our cubic surface is of the type F_2;
(viii) ,, $B_2 < a_0 < B_3$, ,, ,, ,, ,, F_3;
(ix) ,, $B_3 < a_0 < B_4$, ,, ,, ,, ,, F_4;
(x) ,, $B_4 < a_0 < C$, ,, ,, ,, ,, F_5;
(xi) ,, $C < a_0$, ,, ,, ,, ,, F_4.

These conclusions can be derived almost immediately from (ii), (iii), (iv), (v), (vi) respectively, by subjecting our surface to the continuous variation, of the sort considered in § 110, which substitutes for a_0, a_1, a_2, a_3, a_4 respectively:

$$a_0' = a_0-\tfrac{1}{2}t(2a_1-a_2-a_3-a_4), \qquad a_1' = a_1-\tfrac{1}{2}t(2a_1-a_2-a_3-a_4),$$
$$a_2' = a_2, \quad a_3' = a_3, \quad a_4' = a_4, \qquad \text{where } 0 \leqslant t \leqslant 1.$$

Let us finally suppose $a_2 > a_3+a_4$; then we have

$$a_0 > B_2 \quad \text{and} \quad B_1 < B_2 < A < B_3 < B_4 < C,$$

in consequence of (10), (11), and we shall establish that:

(xii) if $B_2 < a_0 < A$, our cubic surface is of the type F_4;
(xiii) ,, $A < a_0 < B_3$, ,, ,, ,, ,, F_3;
(xiv) ,, $B_3 < a_0 < B_4$, ,, ,, ,, ,, F_4;
(xv) ,, $B_4 < a_0 < C$, ,, ,, ,, ,, F_5;
(xvi) ,, $C < a_0$, ,, ,, ,, ,, F_4.

In order to prove (xii), we subject our surface to a continuous deformation by putting, with the usual conventions,

$$a_0' = a_0+t(p-a_0), \qquad a_1' = a_1+t(p-a_1), \qquad a_2' = a_2+t(p-a_2),$$
$$a_3' = a_3+t(q-a_3), \qquad a_4' = a_4+t(q-a_4), \qquad \text{for } 0 \leqslant t \leqslant 1,$$

where $$p = \tfrac{1}{3}(a_0+a_1+a_2), \qquad q = \tfrac{1}{2}(a_3+a_4),$$

so that $$p > 2q > 0.$$

On our hypotheses, for every t of the interval $(0, 1)$ we have

$$B_1' \leqslant B_2' < a_0' < A' < B_3' \leqslant B_4' < C';$$

hence the continuous deformation just considered has the property required in § 110. On the other hand, it leads for $t = 1$ to the surface

$$\frac{1}{p^2}(x_0^3+x_1^3+x_2^3)+\frac{1}{q^2}(x_3^3+x_4^3) = 0,$$

which is of the type F_4 (§ 106) since $p^2/q^2 > 4$: the surface initially considered is therefore of the same type F_4.

The cases (xiii), (xiv), (xv), (xvi) can be respectively derived from (viii), (ix), (x), (xi), by performing the continuous deformation expressed by the formulae

$$a_0' = a_0 - 2t(a_2 - a_3 - a_4), \qquad a_1' = a_1 - t(a_2 - a_3 - a_4),$$
$$a_2' = a_2 - t(a_2 - a_3 - a_4), \qquad a_3' = a_3, \qquad a_4' = a_4, \qquad \text{for } 0 \leqslant t \leqslant 1.$$

In virtue of a remark made in § 110, the results obtained above still hold if in some of the conditions imposed (involving the a_i's) we substitute the sign $=$ for $<$ or $>$; the only limitations in which this is not permitted are those comparing a_0 with A, B_1, B_2, B_3, B_4, C. Thus from (i)–(xvi), taking also into account § 110 and (11), we deduce that:

The complete classification of the non-singular cubic surfaces represented by the equation (1), *where the x_i's are superabundant real homogeneous projective coordinates connected by the identity* (2) *and the λ_i's are positive real numbers satisfying* (8), *can be expressed as follows. If the a_i's are determined by means of* (9), *then the surface referred to is*

of the type F_1, *if*		$a_1+a_2-a_3-a_4 < a_0 < -a_1+a_2+a_3+a_4$;
,, ,, F_2, ,,		$-a_1+a_2+a_3+a_4 < a_0 < a_1-a_2+a_3+a_4$;
,, ,, F_3, ,,	*either*	$a_1-a_2+a_3+a_4 < a_0 < a_1+a_2-a_3+a_4$ *and* $a_3+a_4 \geqslant a_2$,
	or	$a_1+a_2-a_3-a_4 < a_0 < a_1+a_2-a_3+a_4$ *and* $a_3+a_4 \leqslant a_2$;
,, ,, F_4, ,,	*either*	$a_1+a_2-a_3+a_4 < a_0 < a_1+a_2+a_3-a_4$,
	or	$a_1+a_2+a_3+a_4 < a_0$,
	or	$a_1-a_2+a_3+a_4 < a_0 < a_1+a_2-a_3-a_4$;
,, ,, F_5, ,,		$a_1+a_2+a_3-a_4 < a_0 < a_1+a_2+a_3+a_4$.

If in one (and consequently in at least two) of these conditions involving a_0 *the sign $=$ is substituted for* $<$, the surface is *singular* and can be considered at the same time as the limit of a surface F_i and as the limit of a surface F_{i+1} for some convenient value of i ($= 1, 2, 3, 4$); then (§ 109) the singular surface can only have 1, 2, 3, or 4 ordinary double points, and in the first case, which is the general one, *it is necessarily of the type Φ_i* (§ 25). A similar remark can be made in cases II and III.

112. Case II: π_h, π_k, π_l are real and π_m, π_n are conjugate complex (where h, k, l, m, n is an arrangement of the numbers 0, 1, 2, 3, 4). We can suppose λ_h, λ_k, λ_l real and λ_m, λ_n conjugate complex. *If* λ_h, λ_k, λ_l

are not all of the same sign, then the surface F is of the type F_4. In fact, if, for instance, $\lambda_h < 0$, $\lambda_k > 0$, $\lambda_l > 0$, we subject λ_k to a small variation in order to be sure that each determination of the expression (7) has a non-zero real part, and, after that, keeping λ_k, λ_l, λ_m, λ_n unaltered, we reduce λ_h to zero by means of a continuous variation in the interval $(\lambda_h, 0)$; the whole variation is of the sort required in § 110, and leads to a surface which is equianharmonic with 2 real fundamental planes, and therefore (§ 103, II) of the type F_4.

Let us now consider the case in which λ_h, λ_k, λ_l *have the same sign*, which we can suppose to be positive. If we put

$$a_i = |\Re\lambda_i^{-\frac{1}{2}}| \quad (i = 0, 1, 2, 3, 4) \tag{12}$$

we can arrange the suffixes so that the integers m, n are consecutive and

$$a_0 \geqslant a_1 \geqslant a_2 \geqslant a_3 \geqslant a_4 \geqslant 0; \tag{13}$$

moreover, by remembering the remark of § 110, we can suppose that in none of these conditions the equality sign holds, with the single exception given by the obvious equality

$$a_m = a_n, \tag{14}$$

and that each value of the expression

$$\pm a_h \pm a_k \pm a_l \tag{15}$$

is different from zero. On these assumptions, every continuous variation of

$$b = |\Im\lambda_m^{-\frac{1}{2}}| = |\Im\lambda_n^{-\frac{1}{2}}|$$

is of the type considered in § 110, so that the type of F does not depend upon the value of b: in order to determine this type, we can therefore choose b arbitrarily and let the a_i's vary as in § 110. Keeping $b > 0$, we see that the only values of the expression (7) which can vanish during any continuous variation of the a_i's are those given by

$$\pm a_h \pm a_k \pm a_l \pm (a_m + a_n);$$

hence we obtain a deformation of F, of the type required in § 110, by performing any continuous variation of the a_i's during which this last expression does not vanish, letting b vary continuously at the same time in the domain of the positive numbers, in such a way that at the end of the deformation b, but none of the a_i's or of the values of the expression (15), becomes zero.†

We now again define A, B_1, B_2, B_3, B_4, C by means of (11), and

† If we let the a_i's vary keeping b constantly $= 0$, then we should either avoid for the a_i's any values for which (15) becomes zero, or study the behaviour of our surface in the neighbourhood of such values, in correspondence to which the surface would acquire (at least) two double points.

treat separately the four cases which arise for the different values of the consecutive integers m, n $(= 0, 1, 2, 3, 4)$.

Let us first of all suppose $m, n = 0, 1$; then, on account of our previous hypotheses and of § 109, the condition for F to be non-singular is equivalent to supposing a_0 distinct from B_1, and we have only two possibilities:

(i) $a_0 < B_1$, in which case our cubic surface is of the type F_5;
(ii) $a_0 > B_1$, ,, ,, ,, ,, ,, F_4.

In order to prove (i), we derive continuously from F a variable cubic surface $F'(t)$, for which we adopt a similar notation with dashes, putting

$$b' = b(1-t), \qquad a_i' = a_i+t(a_0-a_i) \qquad (i = 0, 1, 2, 3, 4), \quad \text{for } 0 \leqslant t \leqslant 1; \tag{16}$$

such a continuous deformation has the property required in § 110: since the surface $F'(1)$ is a diagonal surface of the type F_5 (§ 102, II), (i) follows. Then (ii) is an immediate consequence, having regard to § 25.

Let us in the second place suppose $m, n = 1, 2$; then the condition for F to be non-singular is equivalent to supposing a_0 distinct from A, B_3, B_4, C, and, since $A < B_3 < B_4 < C$, we have the following 5 possibilities:

(iii) $a_0 < A$, in which case our cubic surface is of the type F_4;
(iv) $A < a_0 < B_3$, ,, ,, ,, ,, ,, F_5;
(v) $B_3 < a_0 < B_4$, ,, ,, ,, ,, ,, F_4;
(vi) $B_4 < a_0 < C$, ,, ,, ,, ,, ,, F_3;
(vii) $C < a_0$, ,, ,, ,, ,, ,, F_4.

(iv) can be easily proved by performing on F (as above) the deformation expressed by (16); whence (iii) and (v) follow at once, on account of § 25. As for (vi), we observe that F can in this case be continuously reduced to a surface of the sort considered in § 107 having $1/16 < \lambda < 1/4$, and therefore actually of the type F_3, by performing on it the deformation (of the type required in § 110) expressed by the formulae

$$b' = b(1-t), \qquad a_0' = a_0, \qquad a_j' = a_j+t(\alpha-a_j) \quad (j = 1, 2, 3, 4), \quad \text{for } 0 \leqslant t \leqslant 1, \qquad \text{where } \alpha = \tfrac{1}{4}(a_1+a_2+a_3+a_4). \tag{17}$$

Finally, in the case (vii), we keep a_1, a_2, a_3, a_4 unaltered and perform on a_0, b the continuous variation

$$b' = b(1-t), \qquad a_0' = a_0/(1-t) \qquad \text{for } 0 \leqslant t \leqslant 1, \tag{18}$$

which is of the sort required in § 110 and reduces F to an equianharmonic surface of the type F_4 (§ 103, II).

We now suppose $m,\ n = 2,\ 3$; then the condition for F to be non-singular is equivalent to supposing a_0 distinct from A, B_1, B_4, C, and 2 cases can be distinguished, according as a_1 is or is not less than a_3+a_4. In any case, owing to (13), (14), we must have $a_0 > A$.

If $a_1 < a_3+a_4$, and therefore $A < B_1 < B_4 < C$, we have the following 4 possibilities:

(viii) $A < a_0 < B_1$, in which case our cubic surface is of the type F_5;
(ix) $B_1 < a_0 < B_4$, ,, ,, ,, ,, ,, F_4;
(x) $B_4 < a_0 < C$, ,, ,, ,, ,, ,, F_3;
(xi) $C < a_0$, ,, ,, ,, ,, ,, F_4.

(viii), (x), (xi) can, in fact, be proved by performing on F the deformations expressed by (16), (17), (18) respectively; then (ix) follows from (viii), on account of § 25.

If $a_1 \geqslant a_3+a_4$, and therefore $B_1 \leqslant A < B_4 < C$, we have the following 3 possibilities:

(xii) $A < a_0 < B_4$, in which case our cubic surface is of the type F_4;
(xiii) $B_4 < a_0 < C$, ,, ,, ,, ,, ,, F_3;
(xiv) $C < a_0$, ,, ,, ,, ,, ,, F_4.

These 3 cases can, in fact, be respectively reduced to (ix), (x), (xi), by performing on F the deformation expressed by

$$a_0' = a_0-t(a_1-a_2-\tfrac{1}{2}a_4), \qquad a_1' = a_1-t(a_1-a_2-\tfrac{1}{2}a_4),$$
$$a_2' = a_2, \quad a_3' = a_3, \quad a_4' = a_4, \qquad \text{for } 0 \leqslant t \leqslant 1,$$

which, taking also into account (14), is easily seen to be of the type required in § 110.

Let us finally suppose $m,\ n = 3,\ 4$; then the condition for F to be non-singular is equivalent to supposing a_0 distinct from A, B_1, B_2, C, and 3 cases can be distinguished.

If $a_2 > a_3+a_4$, and therefore $a_0 > B_2$, $B_1 < B_2 < A < C$, we have only the following 3 possibilities:

(xv) $B_2 < a_0 < A$, in which case our cubic surface is of the type F_2;
(xvi) $A < a_0 < C$, ,, ,, ,, ,, ,, F_3;
(xvii) $C < a_0$, ,, ,, ,, ,, ,, F_4.

In the case (xv), in fact, F can be continuously reduced to a surface of the sort considered in § 106 having $\lambda > 4$, and consequently of the

type F_2, by performing on it the deformation (of the type required in § 110) expressed by the formulae

$$b' = b(1-t), \qquad a_i' = a_i+t(\beta-a_i) \quad (i = 0, 1, 2),$$
$$a_3' = a_4' = a_3 = a_4, \qquad \text{for } 0 \leqslant t \leqslant 1,$$

where β is any number $> a_0-a_1+a_2$ (and therefore also $> 2a_3$). The case (xvii) can be easily examined by means of the transformation (18); and (xvi) follows from (xv), (xvii), having regard to § 25.

If $a_1 < a_3+a_4$, and therefore $a_0 > A$, $A < B_1 < B_2 < C$, we have the following 4 possibilities:

(xviii) $A < a_0 < B_1$, in which case our cubic surface is of the type F_5;
(xix) $B_1 < a_0 < B_2$, ,, ,, ,, ,, ,, F_4;
(xx) $B_2 < a_0 < C$, ,, ,, ,, ,, ,, F_3;
(xxi) $C < a_0$, ,, ,, ,, ,, ,, F_4.

(xviii) can, in fact, be treated like (i) by means of (16); whence (xix) follows, on account of § 25. (xx) can be reduced to (xvi) by performing the transformation

$$a_0' = a_0+4ta_3, \qquad a_1' = a_1+2ta_3, \qquad a_2' = a_2+2ta_3,$$
$$a_3' = a_4' = a_3 = a_4, \qquad \text{for } 0 \leqslant t \leqslant 1.$$

And (xxi), like (vii), can be established by means of (18).

If finally $a_2 < a_3+a_4 < a_1$, and therefore $a_0 > A$, $B_1 < A < B_2 < C$, we have the following 3 possibilities:

(xxii) $A < a_0 < B_2$, in which case our cubic surface is of the type F_4;
(xxiii) $B_2 < a_0 < C$, ,, ,, ,, ,, ,, F_3;
(xxiv) $C < a_0$, ,, ,, ,, ,, ,, F_4.

These 3 cases can, in fact, be respectively reduced to (xix), (xx), (xxi) by means of the deformation

$$\begin{gathered} a_0' = a_0-\tfrac{1}{2}t(2a_1-a_2-a_3-a_4), \qquad a_1' = a_1-\tfrac{1}{2}t(2a_1-a_2-a_3-a_4), \\ a_2' = a_2, \qquad a_3' = a_4' = a_3 = a_4, \qquad \text{for } 0 \leqslant t \leqslant 1, \end{gathered} \tag{19}$$

which is of the type required in § 110. And the stated results continue to hold, even if $a_1 = a_3+a_4$ or $a_2 = a_3+a_4$.

From (i)–(xxiv), taking also into account § 110 and (11), we deduce that:

If the Sylvester pentahedron of the non-singular cubic surface F represented by (1), (2) *has* 3 *faces π_h, π_k, π_l real and* 2 *faces π_m, π_n conjugate complex, and, moreover, λ_h, λ_k, λ_l are real and positive, we define the a_i's*

by means of (12), *and arrange the suffixes so that* m, n *are consecutive and the conditions* (13) *hold. Then the surface* F *is*

of the type F_2, *if*

$$m, n = 3, 4 \quad \textit{and } a_1-a_2+a_3+a_4 < a_0 < a_1+a_2-a_3-a_4;$$

of the type F_3, *if either*

$$\left.\begin{array}{ll} & m, n = 1, 2 \\ or & m, n = 2, 3 \end{array}\right\} \textit{ and } a_1+a_2+a_3-a_4 < a_0 < a_1+a_2+a_3+a_4,$$

$$\text{,,}\quad m, n = 3, 4 \quad \text{,,}\quad a_1-a_2+a_3+a_4 < a_0 < a_1+a_2+a_3+a_4, \quad a_2 \leqslant a_3+a_4,$$

$$\text{,,}\quad m, n = 3, 4 \quad \text{,,}\quad a_1+a_2-a_3-a_4 < a_0 < a_1+a_2+a_3+a_4, \quad a_2 \geqslant a_3+a_4;$$

of the type F_4, *if either*

$$m, n = 0, 1 \quad \textit{and } a_0+a_1 > a_2+a_3+a_4,$$

$$or\quad m, n > 0 \quad \text{,,}\quad a_0 > a_1+a_2+a_3+a_4,$$

$$\text{,,}\quad m, n = 1, 2 \quad \text{,,}\quad a_1+a_2-a_3+a_4 < a_0 < a_1+a_2+a_3-a_4,$$

$$\text{,,}\quad m, n = 1, 2 \quad \text{,,}\quad a_0 < a_1+a_2-a_3-a_4,$$

$$\text{,,}\quad m, n = 2, 3 \quad \text{,,}\quad -a_1+a_2+a_3+a_4 < a_0 < a_1+a_2+a_3-a_4, \quad a_1 \leqslant a_3+a_4,$$

$$\text{,,}\quad m, n = 2, 3 \quad \text{,,}\quad a_1+a_2-a_3-a_4 < a_0 < a_1+a_2+a_3-a_4, \quad a_1 \geqslant a_3+a_4,$$

$$\text{,,}\quad m, n = 3, 4 \quad \text{,,}\quad -a_1+a_2+a_3+a_4 < a_0 < a_1-a_2+a_3+a_4, \quad a_1 \leqslant a_3+a_4,$$

$$\text{,,}\quad m, n = 3, 4 \quad \text{,,}\quad a_1+a_2-a_3-a_4 < a_0 < a_1-a_2+a_3+a_4, \quad a_1 \geqslant a_3+a_4;$$

of the type F_5, *if either*

$$m, n = 0, 1 \quad \textit{and } a_0+a_1 < a_2+a_3+a_4,$$

$$or\quad m, n = 1, 2 \quad \text{,,}\quad a_1+a_2-a_3-a_4 < a_0 < a_1+a_2-a_3+a_4,$$

$$\left.\begin{array}{ll} \text{,,} & m, n = 2, 3 \\ \text{,,} & m, n = 3, 4 \end{array}\right\} \text{ ,,}\quad a_1+a_2-a_3-a_4 < a_0 < -a_1+a_2+a_3+a_4;$$

and no other cases can arise.

113. CASE III: π_l IS REAL, π_h, π_k AND π_m, π_n ARE CONJUGATE COMPLEX. We can suppose λ_l real and positive, λ_h, λ_k and λ_m, λ_n conjugate complex, so that, if we still define the a_i's by means of (12), we have

$$a_l > 0, \qquad a_m = a_n, \qquad a_h = a_k. \tag{20}$$

Moreover, if we put

$$b = |\Im\lambda_m^{-\frac{1}{2}}| = |\Im\lambda_n^{-\frac{1}{2}}|, \qquad c = |\Im\lambda_h^{-\frac{1}{2}}| = |\Im\lambda_k^{-\frac{1}{2}}|,$$

by subjecting the λ_i's to a small variation, inducing on F a deformation of the type considered in § 110, we can always arrange to have

$$b > 0, \qquad c > 0;$$

then the only values of the expression (7) which can vanish during any continuous variation of the a_i's are those given by

$$\pm a_l \pm (a_m + a_n) \pm (a_h + a_k).$$

Therefore we obtain a deformation of F of the type required in § 110, by performing any continuous variation of the a_i's during which the last expression does not vanish, letting at the same time b, c vary continuously in the domain of the positive numbers, in such a way that at the end of the deformation both b and c, but none of the expressions $a_0 a_1 a_2 a_3 a_4$, $a_l \pm (a_m + a_n)$, $a_l \pm (a_h + a_k)$, become null.

If we arrange the suffixes so that m, n and h, k are consecutive and the conditions (13) hold, we see that l can only have an even value. We shall now study the various cases that thus arise; and we can make the further assumption, which involves no restriction, that in the conditions (13) only two equality signs hold, given by (20).

Let us first of all suppose $l = 0$; then the condition for F to be non-singular is equivalent to supposing a_0 distinct from A, C, and we have 3 possibilities:

(i) $a_0 < A$, in which case our cubic surface is of the type F_4,
(ii) $A < a_0 < C$, ,, ,, ,, ,, ,, F_3;
(iii) $C < a_0$, ,, ,, ,, ,, ,, F_2.

In order to prove (i), we derive continuously from F a variable cubic surface $F'(t)$, for which we adopt a similar notation with dashes, putting

$$b' = b(1-t), \qquad c' = c(1-t), \qquad a_0' = a_0 - t(a_0 - a_1),$$

$$a_1' = a_2' = a_1 = a_2, \qquad a_3' = a_4' = a_3 = a_4, \qquad \text{for } 0 \leqslant t \leqslant 1;$$

such a continuous deformation has the property required in § 110: since $F'(1)$ is a surface of the sort considered in § 106 (with $\lambda > 4$), which is of the type F_4, (i) follows. In the case (iii), we keep a_1, a_2, a_3, a_4 unaltered and perform on a_0, b, c the continuous variation

$$b' = b(1-t), \qquad c' = c(1-t), \qquad a_0' = a_0/(1-t), \qquad \text{for } 0 \leqslant t \leqslant 1,$$

which is of the kind considered in § 110 and reduces F to an equianharmonic surface of the type F_2 (§ 103, III). Hence (ii) can be deduced from (i), (iii), on account of § 25.

Let us in the second place suppose either $l = 2$ *or* $l = 4$; then the

condition for F to be non-singular is equivalent to supposing a_0 distinct from B_1, and we have only 2 possibilities:

(iv) $a_0 < B_1$, in which case our cubic surface is of the type F_3;

(v) $a_0 > B_1$, „ „ „ „ „ F_4.

(iv) can, in fact, be proved by performing on F the deformation

$$b' = b(1-t), \qquad c' = c(1-t), \qquad a'_i = a_i + t(a_0 - a_i)$$
$$(i = 0, 1, 2, 3, 4), \qquad \text{for } 0 \leqslant t \leqslant 1,$$

giving (for $t = 1$) a diagonal surface of the type F_3 (§ 102, III). And (v) can be proved by performing on F the deformation

$$b' = b(1-t), \qquad c' = c(1-t), \qquad a'_2 = a_2 - t(a_2 - a_3),$$
$$a'_0 = a'_1 = a_0 = a_1, \qquad a'_3 = a'_4 = a_3 = a_4, \qquad \text{for } 0 \leqslant t \leqslant 1,$$

if $l = 2$, or the deformation

$$b' = b(1-t), \qquad c' = c(1-t), \qquad a'_2 = a'_3 = a_2 - t(a_2 - a_4),$$
$$a'_0 = a'_1 = a_0 = a_1, \qquad a'_4 = a_4, \qquad \text{for } 0 \leqslant t \leqslant 1,$$

if $l = 4$, leading in both cases to a surface of the sort considered in § 106 (with $0 < \lambda < 4/9$), which is of the type F_4.

From (i)–(v), taking also into account § 110 and (11), we deduce that: *If only one, π_l say, of the faces of the Sylvester pentahedron of the non-singular cubic surface F represented by* (1), (2) *is real, we can suppose λ_l real and positive and arrange the suffixes so that, defining the a_i's by means of* (12), *the conditions* (13) *hold. Then the surface F is*

of the type F_2, if	$l = 0$	*and*	$a_0 > a_1+a_2+a_3+a_4$;
„ „ F_3, „ *either*	$l = 0$	„	$a_1+a_2-a_3-a_4 < a_0 < a_1+a_2+a_3+a_4$,
or	$l > 0$	„	$a_0+a_1 < a_2+a_3+a_4$;
„ „ F_4, „ *either*	$l = 0$	„	$a_0 < a_1+a_2-a_3-a_4$,
or	$l > 0$	„	$a_0+a_1 > a_2+a_3+a_4$;

and no other cases can arise.

114. If a non-singular cubic surface F is non-generic, either it is cyclic (in particular equianharmonic) or it is representable by a canonical equation of the form (6) or (9) of §§ 91, 93 (cf. § 94). The type of F is given at once in the first case by §§ 103, 104; in the second case it can be deduced without difficulty from §§ 111–13 by noticing that—when $\epsilon \to 0$—the surface F' represented by

$$\left(\frac{1}{\epsilon} - \lambda_0\right)x_0^3 + x_1^3 + x_2^3 + x_3^3 - \frac{1}{\epsilon}(x_0 + \epsilon\lambda_1 x_1 + \epsilon\lambda_2 x_2 + \epsilon\lambda_3 x_3)^3 = 0 \qquad (21)$$

or by

$$\left(2\lambda+\frac{1}{\epsilon^2}\right)x_0^3+x_1^3+x_2^3-\frac{1}{2\epsilon^2}(x_0+\epsilon^2\mu x_1+\epsilon^2 x_2+\epsilon x_3)^3- \\ -\frac{1}{2\epsilon^2}(x_0+\epsilon^2\mu x_1+\epsilon^2 x_2-\epsilon x_3)^3 = 0 \tag{22}$$

tends to the surface F represented by one of those equations (6) or (9) respectively: so that, if ϵ is real, non-zero, and sufficiently near to 0, the type of F is the same as that of F'. Each of the equations (21), (22) can be reduced to the canonical form

$$\sum_{i=0}^{4}\lambda_i' x_i'^3 = 0, \qquad \text{with} \quad \sum_{i=0}^{4} x_i' = 0,$$

by putting

$$x_0':x_1':x_2':x_3':x_4' = x_0:\epsilon\lambda_1 x_1:\epsilon\lambda_2 x_2:\epsilon\lambda_3 x_3:-(x_0+\epsilon\lambda_1 x_1+\epsilon\lambda_2 x_2+\epsilon\lambda_3 x_3),$$

$$\lambda_0':\lambda_1':\lambda_2':\lambda_3':\lambda_4' = \epsilon^2(1-\lambda_0\epsilon):1/\lambda_1^3:1/\lambda_2^3:1/\lambda_3^3:\epsilon^2,$$

or, respectively,

$$x_0':x_1':x_2':x_3':x_4'$$
$$= 2x_0:2\epsilon^2\mu x_1:2\epsilon^2 x_2:-(x_0+\epsilon^2\mu x_1+\epsilon^2 x_2+\epsilon x_3):-(x_0+\epsilon^2\mu x_1+\epsilon^2 x_2-\epsilon x_3),$$

$$\lambda_0':\lambda_1':\lambda_2':\lambda_3':\lambda_4' = \epsilon^4(1+2\lambda\epsilon^2):1/\mu^3:1:4\epsilon^4:4\epsilon^4;$$

simple considerations give then the type of F', and in particular show that, if $|\epsilon|$ is sufficiently small, F' (and therefore also F) is certainly never of the type F_1.

Taking also into account §§ 103, 104, 111–13, we can therefore say that:

A real cubic surface with 27 *distinct real lines has always a well-determined Sylvester pentahedron, consisting of* 5 *distinct real planes, no* 4 *of which are dependent.*

APPENDIX I

THE DEGENERATION OF A NON-SINGULAR CUBIC SURFACE INTO A PLANE AND A QUADRIC

LET us consider (in the ordinary space) a non-singular cubic surface F, tending to a non-singular quadric Q and a plane π, such that Q, π have in common an irreducible conic $\mathfrak{C}$, which intersects F in 6 points having distinct limits P_i ($i = 1, 2,..., 6$). The envelope of the 12th class determined by F has then a limit which breaks up into the envelope of Q, the envelope of $\mathfrak{C}$ counted twice, and the 6 nets of planes of centres P_i;† and each line r_0 which is the limit of a line r of F must be the axis of a pencil having multiplicity at least 2 for the degenerating envelope (§ 1). r_0 is certainly not a tangent of $\mathfrak{C}$, since otherwise r would intersect Q in 2 points tending to the point O of contact of r_0 and $\mathfrak{C}$, so that the limit of the intersection of F and Q would touch π at O, and 2 of the points P_i would coincide with O. It follows that r_0 can only be one of the 15 lines $c'_{ij} = P_i P_j$ ($i, j = 1, 2,..., 6$; $i \neq j$), or one of the 6 generators a'_i of one system of Q through the points P_i, or one of the 6 generators b'_i of the other system through the same points; and, by an argument already used in § 2, we see that none of these 27 lines can be the limit of 2 distinct lines of F, so that each of them is in fact the limit of one of the 27 lines of F.

The incidence relations among the latter can easily be obtained, by means of considerations similar to those developed in § 5; and we can say that:

When the cubic surface F degenerates into $Q+\pi$, 12 lines of F—forming a double-six—have for their limits lines of Q, and the 15 others have their limits on π. More precisely, the 27 lines of F can be represented in relation to such a double-six by the Schläfli notation (§ 22), so that

$$\lim a_i = a'_i, \qquad \lim b_i = b'_i, \qquad \lim c_{ij} = c'_{ij} \quad (i, j = 1, 2,..., 6;\ i \neq j),$$

where a'_i, b'_i, c'_{ij} are the lines defined above.

This result leads in a natural way to the consideration of the double-sixes and to the Schläfli notation, giving at the same time another graphical representation for the 27 lines of a cubic surface; which, however, is less simple and useful than that studied at length in this work: it cannot, for instance, apply in the real domain to the surfaces F_5, as these do not contain any self-conjugate double-six consisting of two self-conjugate sextuplets (§ 33).

† Cf., for instance, B. Segre, loc. cit. in § 2.

APPENDIX II

THE DEGENERATION OF A NON-SINGULAR CUBIC PRIMAL OF [4] INTO THREE PRIMES

LET us consider in [4] a non-singular cubic primal V, degenerating into a primal V_0 consisting of three independent primes Π_1, Π_2, Π_3. If i, h, k are the numbers 1, 2, 3 taken in any order, we consider the intersection of V with the plane $\pi_i = \Pi_h \Pi_k$, and suppose that its limit for $V \to V_0$ is a non-singular plane cubic $\mathfrak{C}_i$, which intersects the line p common to π_1, π_2, π_3 in three distinct points P_1, P_2, P_3 (independent of i). On V there is an ∞^2 system Σ of lines; and, taking into account sections I, II, we can easily obtain its limit Σ_0 and deduce the properties of the former from those of the latter. We confine ourselves to establishing in this way a few results of an enumerative character,† leaving to the reader further investigations on the subject and applications to questions of reality.

We notice, first of all, that Σ_0 consists of the totality of the lines (of V_0) intersecting two of the curves $\mathfrak{C}_1$, $\mathfrak{C}_2$, $\mathfrak{C}_3$, in two generally distinct points; these, however, can be non-distinct, if they coincide with one and the same point P_i, in which case the corresponding line of Σ_0 must belong to one of the three pencils determined by the tangents of two of the curves $\mathfrak{C}_1$, $\mathfrak{C}_2$, $\mathfrak{C}_3$ at such a point. Two incident lines of Σ_0 are the limits of two incident lines of V, if they either belong to different spaces Π_i or have in common a point lying outside the curves $\mathfrak{C}_i$ (§ 5).

Through each point of V there are 6 *lines of* Σ; this corresponds to the fact that we have 6 lines of Σ_0 through a generic point of each space Π_i, given by the 6 lines through this point incident to $\mathfrak{C}_h$, $\mathfrak{C}_k$ in points distinct from P_1, P_2, P_3. If we consider a generic line r_0 of Σ_0, intersecting $\mathfrak{C}_h$, $\mathfrak{C}_k$ at R_h, R_k respectively, the lines of Σ_0 incident with it constitute the two cubic cones projecting $\mathfrak{C}_i$ from R_h, R_k, and also the locus of the lines of Π_i incident to $\mathfrak{C}_h$, $\mathfrak{C}_k$, r_0 in three generally distinct points. The two cones are elliptic and have no generator in common; each of them has three generators in common with the locus just considered, which is a ruled surface of order 9 and genus 5. It follows that *the lines of V, incident with a generic line of V itself, constitute a ruled surface of order* 15 *and genus* 11.

The envelope of the common tangent planes of $\mathfrak{C}_h$, $\mathfrak{C}_k$ is obviously of class $6.6 = 36$; it has three bitangent planes (given by the planes joining the tangents of C_h, C_k at one of their common points P_1, P_2, P_3), 108 inflexional tangent planes (given by the planes through an inflexional tangent of one of the curves C_h, C_k which also touch the other), and is of genus 34.‡ Hence, in virtue of Cayley's formulae,|| the edge of regression of such an envelope, Γ_i say, is of order 90, rank 30, and has 216 cusps and no inflexion. The developable D_i (of order 30) formed by the tangents of Γ_i has on π_h 12 generators, given by the tangent lines of $\mathfrak{C}_h$ intersecting this curve further at one of the points P_1, P_2, P_3; it is easily seen that Γ_i and $\mathfrak{C}_h$ touch such a generator at the same point, the two curves

† Already directly proved by G. Fano, 'Ricerche sulla varietà cubica generale dello spazio a quattro dimensioni e sopra i suoi spazi pluritangenti', *Ann. di Mat.* (III), vol. 10 (1904), pp. 251–85, §§ 7–11.

‡ All these results appear almost evident in the dual form.

|| Cf., for instance, F. Severi, *Trattato di geometria algebrica* (Bologna, Zanichelli, 1926), p. 139.

being therefore tangent at this point to the curve Γ_k of suffix $k \neq i, h$. It follows that

The cubic primal V contains a developable D, locus of the points of V such that two (at least) among the 6 lines of V through them coincide, these coinciding lines being precisely the generators of D; the developable D is of order 90 and genus 136.

In fact, when $V \to V_0$, D breaks up into the 3 developables D_i, each of order 30 and genus 34, any 2 of which have 12 generators in common. As the edge of regression of D breaks up in the limit into the 3 curves Γ_i (each of order 90), we see, moreover, that

The edge of regression of D, namely, the locus Γ of the points of V such that three (at least) among the 6 lines of V through them coincide, is of order 270.

Since Γ (like each of the curves Γ_i) has no inflexions, in virtue of Veronese's formulae,† we have that

Γ has 720 cusps, its 2nd rank is equal to 180, and its class is equal to 540.

When $V \to V_0$, 648 among the cusps of Γ tend to the 3×216 cusps of the curves Γ_i ($i = 1, 2, 3$), and the remaining 72 tend in pairs to the above-considered $3 \times 12 = 36$ points of contact of 2 of these curves. The osculating planes of Γ, namely, the tangent planes of D, are the planes of [4] which intersect V along three lines one of which has to be counted twice, and generate a primal of order 180; in the limit they become the osculating planes of the curves Γ_i, constituting 3 envelopes of class 36, and, moreover, the planes of the 36 pencils determined by the planes touching 2 of the developables D_i along a common generator, each pencil having to be counted twice.

† Cf., for instance, F. Severi, op. cit., p. 142.

APPENDIX III

ON SINGULAR SURFACES OF A PENCIL OF CUBIC SURFACES

The results (i)–(v) stated at the beginning of § 25 can be proved as follows. By introducing in [3] appropriate non-homogeneous coordinates (x, y, z) vanishing at Q, we can represent the cubic surface Φ by the equation

$$\phi(x,y,z) = \phi_2(x,y,z)+\phi_3(x,y,z) = 0,$$

where ϕ_3 is a cubic form, and $\phi_2(x,y,z) = x^2+y^2+z^2$ or $\phi_2(x,y,z) = x^2+y^2$ according as the cone K is irreducible or reducible. We notice that in the second case ϕ_3 must contain z^3 with a non-vanishing coefficient, since the six lines of Φ through Q (represented by $\phi_2 = \phi_3 = 0$) are supposed to be distinct.

A pencil of cubic surfaces containing Φ is represented by an equation of the form

$$F(x,y,z;\lambda) = \phi(x,y,z)+\lambda\psi(x,y,z) = 0,$$

$\phi = \psi = 0$ being the equations of the base curve $\mathfrak{C}$. If we interpret (x,y,z,λ) as non-homogeneous coordinates in [4], we see that the number of times that Φ counts among the 32 singular surfaces of the pencil is equal to the multiplicity m of intersection of the following primals of [4] at the origin $O(0,0,0,0)$:

$$\frac{\partial F}{\partial x} = 0, \qquad \frac{\partial F}{\partial y} = 0, \qquad \frac{\partial F}{\partial z} = 0, \qquad x\frac{\partial F}{\partial x}+y\frac{\partial F}{\partial y}+z\frac{\partial F}{\partial z}-3F = 0. \qquad (I)$$

In case (i) the four primals (I) go simply through O, having at this point four linearly independent tangent primes, whence $m = 1$; but in case (ii) $m \geqslant 2$, since the fourth primal (I) has then a double point at O. In cases (iii) or (iv) respectively the two first and either the last or the last but one of the primals (I) go simply through O, with independent tangent primes, so that they intersect along a curve going through O with a single linear branch; m is therefore the multiplicity of intersection at O of this branch and the remaining primal (I), and an easy calculation shows that $m = 2$ or $m = 3$ in cases (iii) or (iv) respectively. In case (v) the last two primals (I) have both a double point at O, so that $m \geqslant 4$.

The above-indicated process can be used, with obvious modifications, to extend the results (i)–(v) of § 25 to pencils of surfaces of [3] of any order.

INDEX

(Numbers refer to paragraphs)